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PART II

# MECHANICS OF COMPOSITE MATERIALS

## PART II — THEORETICAL ASPECTS

*STEPHEN W. TSAI*

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## FOREWORD

This report covers the second portion of the notes prepared for a seminar, "Mechanics of Composite Materials," presented at the Air Force Materials Laboratory in April and May 1966. The work was initiated under Project No. 7340, "Nonmetallic and Composite Materials," Task 734003, "Structural Plastics and Composites." The seminar consisted of Part I - Introduction, and Part II - Mathematical Theory.

The manuscript of this report was released by the author in June 1966 for publication as an RTD Technical Report.

This technical report has been reviewed and is approved.

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## ABSTRACT

This report covers some of the principles of the mechanics relevant to the description c composite materials. The contents of these notes may provide useful information for th understanding of current publications and reports related to composite materials.

Emphasis is placed on the use of indicial notation and operations. The rules governing th use of the contracted notation are also outlined. The generalized Hooke's law and its trans formation properties, material symmetries, and engineering constants are also discussed The plane strain and plane stress problems are discussed in detail. Finally, the elastic modul of laminated anisotropic materials, and the strength of both unidirectional and laminated com posites are covered.

# TABLE OF CONTENTS

Each tensor can also be arranged in a matrix form. The indices of each tensorial component can be associated with a specific position in the matrix. For example, a vector $a_i$ can be expressed by A, or a column or row matrix form as follows:

$$a_i = A = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} \text{ or } (a_1 \quad a_2 \quad a_3) \tag{2}$$

where i = 1, 2, 3, which indicates that the space is three-dimensional. In 2-space, i = 1, 2, Equation 2 becomes

$$a_i = A = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix} \text{ or } (a_1 \quad a_2) \tag{3}$$

For a second-rank tensor $a_{ij}$, the corresponding matrix forms for 2-space and 3-space are, respectively:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \tag{4}$$

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \tag{5}$$

For a fourth-rank tensor, the matrix form in 2-space is:

$$a_{ijkl} = \begin{bmatrix} a_{1111} & a_{1122} & a_{1112} & a_{1121} \\ a_{2211} & a_{2222} & a_{2212} & a_{2221} \\ a_{1211} & a_{1222} & a_{1212} & a_{1221} \\ a_{2111} & a_{2122} & a_{2112} & a_{2121} \end{bmatrix} \tag{6}$$

A fourth-rank tensor in 3-space contains 81 components which can be arranged in a 9 x 9 matrix.

Matrices contain an array of numbers. The numbers arranged in appropriate positions in a matrix may represent the components of a tensor. This does not mean that tensors and matrices are identically equal. The components of a matrix may be arbitrary and completely unrelated. But the components of a tensor, whether in the indicial notation or the matrix form, are not arbitrary. The components are governed by a set of rules, called the transformation equations. Therefore, a tensor can be considered as a special type of matrix. Tensors are in fact defined by the transformation equations. If a set of numbers satisfy certain transformation equations they are by definition components of a tensor. For each tensorial rank, there is a corresponding transformation equation. Thus, the coordinate transformation is a basic feature of tensors.

In a simple geometric term, a coordinate transformation involves the rotation of one reference coordinate system relative to another. The relation between the old and new coordinates, designated by $x_i$ and $x_i'$, are, for $i = 1, 2,$ and $3$ as follows:

$$
\begin{aligned}
x_1' &= t_{11} x_1 + t_{12} x_2 + t_{13} x_3 \\
x_2' &= t_{21} x_1 + t_{22} x_2 + t_{23} x_3 \\
x_3' &= t_{31} x_1 + t_{32} x_2 + t_{33} x_3
\end{aligned}
\tag{7}
$$

where $t_{ij}$ are direction cosines between the i-axis in the new system and the j-axis of the old one. Equation 7 may be written in a matrix form:

$$
\begin{bmatrix} x_1' \\ x_2' \\ x_3' \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}
\tag{8}
$$

where the usual rule of matrix multiplication applies and this rule can be expressed in the indicial notation as follows:

$$
\begin{aligned}
x_1' &= \sum_{j=1}^{3} t_{1j} x_j = t_{11} x_1 + t_{12} x_2 + t_{13} x_3 \\
x_2' &= \sum_{j=1}^{3} t_{2j} x_j = t_{21} x_1 + t_{22} x_2 + t_{23} x_3 \\
x_3' &= \sum_{j=1}^{3} t_{3j} x_j = t_{31} x_1 + t_{32} x_2 + t_{33} x_3
\end{aligned}
\tag{9}
$$

Two useful conventions of the indicial notation can now be introduced:

1) **Range Convention:**

Unrepeated index (free index) takes all the values 1, 2, . . . , n, where n is the dimension of the space.

2) **Summation Convention:**

Repeated index (dummy index) calls for the summation with respect to that index with the range of summation determined by the range convention.

With these conventions, Equation 9 can be written as:

$$
x_i = t_{ij} x_j
\tag{10}
$$

where i is the free index and is equal to 1, 2, and 3 in accordance with the range convention, j is the dummy index, as it is repeated in the right-hand side of Equation 10, and a summation of j with the same range as i, i.e, 1, 2, and 3, is implied. The use of summation convention replaces the summation sign (capital Sigma) in Equation 9. Both range and summation conventions are applicable to all square matrices (because all indices must have the same range) and are therefore not limited to tensors.

Equations 7, 8, and 10 are identical equations and it is quite clear that the use of the indicial notation introduces a significant simplification.

TRANSFORMATION MATRIX

As stated earlier, $t_{ij}$, which for convenience is designated T, are direction cosines between the old and new coordinate systems. Knowing the angle of rotation between the two coordinate systems, the direction cosines can be determined immediately. The components of the transformation matrix T for a rotation about the 3-axis can be derived from Figure 1 and are listed in Table 1:

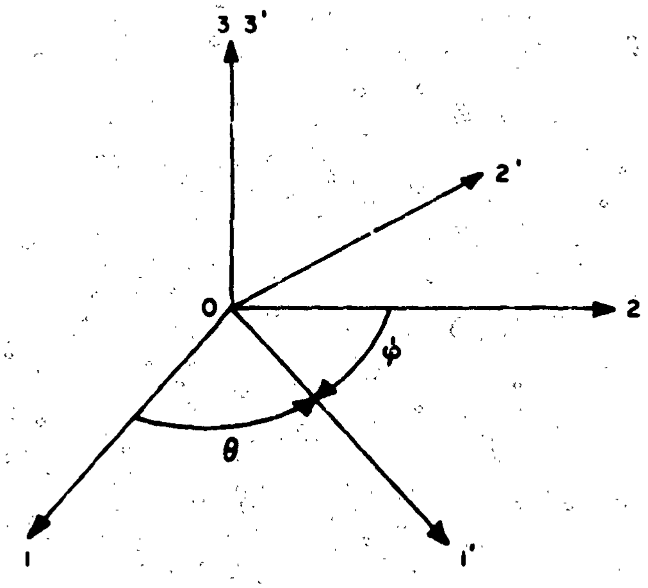


Figure 1. Coordinate Rotation

TABLE I

T-MATRIX FOR ROTATION ABOUT 3-AXIS

| Components | Angles | Direction Cosines |
|---|---|---|
| $t_{11}$ | $1'01 = \theta$ | $\cos\theta$ |
| $t_{12}$ | $1'02 = \phi$ | $\sin\theta$ |
| $t_{13}$ | $1'03 = 90°$ | 0 |
| $t_{21}$ | $2'01 = 90° + \theta$ | $-\sin\theta$ |
| $t_{22}$ | $2'02 = \theta$ | $\cos\theta$ |
| $t_{23}$ | $2'03 = 90°$ | 0 |
| $t_{31}$ | $3'01 = 90°$ | 0 |
| $t_{32}$ | $3'02 = 90°$ | 0 |
| $t_{33}$ | $3'03 = 0$ | 1 |

If $m = \cos\theta$ and $n = \sin\theta$, the T-matrix above can be arranged in a matrix form as follows:

$$T = t_{ij} = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{11}$$

For a rotation in the opposite direction of that shown in Figure 1, $\theta$ is negative, and the transformation matrix, designated $T^-$, is as follows:

$$T^- = \begin{bmatrix} m & -n & 0 \\ n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{12}$$

The components of the T-matrix for any other rotation can be derived similarly from the angles listed in Table I. The subscripts of the $t_{ij}$ refer to the new and old axes, i.e., the angle for the direction cosine is that between the i-th axis of the new coordinates and the j-th axis of the old coordinates.

There are a number of features of the T-matrix, as follows:

1) When $\theta = 0$.

$$T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{13}$$

This is called the identity transformation, for which $x_i' = x_i$. The new and the old coordinates are identical.

2) When $\theta$ = 180°,

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{14}$$

This is called the central inversion, for which $x_1' = -x_1$, $x_2' = -x_2$, and $x_3' = x_3$.

3) A proper rotation is defined by a T-matrix with its determinant equal to +1, i.e., $|T| = +1$. For a 3 x 3 determinant, its numerical value is the sum of the following products:

$$\begin{array}{l} t_{11}\ t_{22}\ t_{33} \\ t_{12}\ t_{23}\ t_{31} \\ t_{21}\ t_{32}\ t_{13} \\ -t_{31}\ t_{22}\ t_{13} \\ -t_{21}\ t_{12}\ t_{33} \\ -t_{11}\ t_{23}\ t_{32} \end{array} \tag{15}$$

The determinants of the T-matrix in Equations 11, 12, 13, and 14 are equal to +1, and are thus proper rotations. Geometrically speaking, all of these transformations preserved the right-hand system of coordinates, as shown in Figure 1. This system satisfies the right-hand rule, when it is applied to the coordinate axes, by rotating the 1-axis toward the 2-axis with the 3-axis as the advancing screw.

If a transformation changes the right-hand system of coordinates into a left-hand system, this is called an improper rotation, for which $|T| = -1$. An example of this transformation is the reflection of the 1-axis, such that $x_1' = -x_1$, $x_2' = x_2$, and $x_3' = x_3$. The T-matrix becomes:

$$T = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{16}$$

Note that $|T| = -1$, which indicates that the rotation is improper and the new coordinates $x_i'$ are now a left-hand system. This is shown in Figure 2.

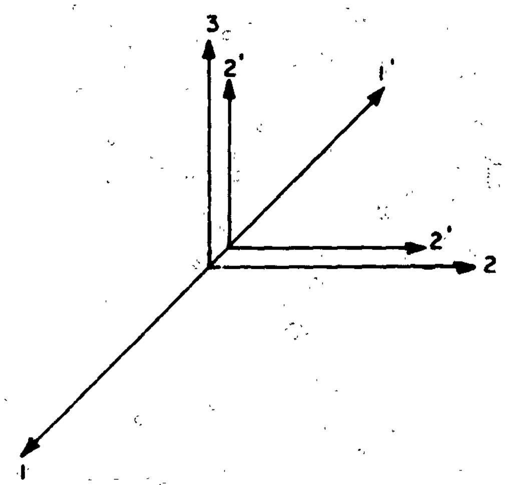


Figure 2. Improper Rotations

4) The T-matrix must also satisfy a geometric relation known as the orthogonality condition. If the coordinate axes are orthogonal to one another, the components of the T-matrix, which are the direction cosines must satisfy

$$t_{ik}\, t_{jk} = \delta_{ij} \tag{17}$$

where $\delta_{ij}$ is the Kronecker delta, and

$$\begin{aligned} \delta_{ij} &= 1 \text{ when } i = j \\ \delta_{ij} &= 0 \text{ when } i \neq j \end{aligned} \tag{18}$$

The Kronecker delta in a matrix form is:

$$\delta_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{19}$$

which is also a unit matrix. The use of Kronecker delta is a very important tool in tensor operations.

TENSOR CALCULUS

Three salient features of the indicial notation mentioned thus far are the range convention, the summation convention and the Kronecker delta. When they are applied simultaneously, one can show that

$$\begin{aligned} \delta_{ij}\, x_j &= x_i \\ \delta_{ij}\, a_{ij} &= a_{ii} \\ \delta_{ij}\, a_{ik}\, a_{kj} &= a_{ik}\, a_{ki} \end{aligned} \tag{20}$$

$$\delta_{ii} = 2 \text{ for } i = 1, 2, \qquad = 3 \text{ for } i = 1, 2, 3 \tag{20}$$

Another feature of the indicial notation is a direct correspondence between the number of free indices, k (not the dummy index, because $a_{ii}$ is a scalar, $a_{iij}$, a vector), and the tensorial rank. The number of components, N, in each term is determined by Equation 1, where k is the number of free indices and n is the number of dimensions.

Finally, in this notation, a comma represents differentiation, as follows:

$$\frac{\partial a_i}{\partial x_j} = a_{i,j} \tag{21}$$

By using the summation convention, one obtains

$$\begin{aligned} a_{i,i} &= \frac{\partial a_i}{\partial x_i} \\ &= \frac{\partial a_1}{\partial x_1} + \frac{\partial a_2}{\partial x_2} + \frac{\partial a_3}{\partial x_3} \\ &= \nabla \cdot a \\ &= \text{div } a \end{aligned} \tag{22}$$

where, in the last two steps, the conventional vector notation of divergence is used. Similarly, for a scalar function A,

$$\begin{aligned} A_{,i} &= \frac{\partial A}{\partial x_i} \\ &= \frac{\partial A}{\partial x_1} + \frac{\partial A}{\partial x_2} + \frac{\partial A}{\partial x_3} \\ &= \nabla A \\ &= \text{grad } A \end{aligned} \tag{23}$$

where, in the last two steps, the conventional notation for gradient is used.

Finally,

$$
\begin{aligned}
A_{,ii} &= \frac{\partial^2 A}{\partial x_i \partial x_i} \\
&= \frac{\partial^2 A}{\partial x_1^2} + \frac{\partial^2 A}{\partial x_2^2} + \frac{\partial^2 A}{\partial x_3^2} \\
&= \mathrm{div}\,(\nabla A) \\
&= \nabla^2 A
\end{aligned}
\tag{24}
$$

which is the Laplacian operator. Since the tensorial rank can be determined by the number of free indices, $a_{i,i}$ and $A_{,ii}$ are scalars; and $A_{,i}$, a vector. The fact that the tensorial rank can be determined by observation is a feature of the indicial notation that does not exist in the conventional vector notation.

The divergence theorem can be written as follows:

$$\int_A a_i n_i \, dA = \int_V a_{i,i} \, dV \tag{25}$$

where V = volume, A = surface, $n_i$ = exterior normal to A. Substituting $a_i = A_{,i}$ into Equation 25

$$\int_A A_{,i} n_i \, dA = \int_V A_{,ii} \, dV \tag{26}$$

This relation can be used, for example, in the derivation of the Fourier heat conduction equation. Equation 25 can also be generalized to a vector equation by using $\sigma_{ji}$ in place of $a_i$

$$\int_A \sigma_{ji} n_i \, dA = \int_V \sigma_{ji,i} \, dV \tag{27}$$

This relation will be used in the derivation of the equilibrium equation.

## STRESS TENSOR

Stress is a measure of the internal forces in a continuous medium induced by surface forces applied to a body. The relationship can be defined by

$$T_i = \sigma_{ji} n_j \tag{28}$$

where $T_i$ = surface traction, $\sigma_{ji}$ = stress tensor, and $n_i$ = directional cosines of the surface on which $T_i$ acts. Equation 28 in 2-space is:

$$
\begin{aligned}
T_1 &= \sigma_{11} n_1 + \sigma_{21} n_2 \\
T_2 &= \sigma_{12} n_1 + \sigma_{22} n_2
\end{aligned}
\tag{29}
$$

The relations can be seen in Figure 3.

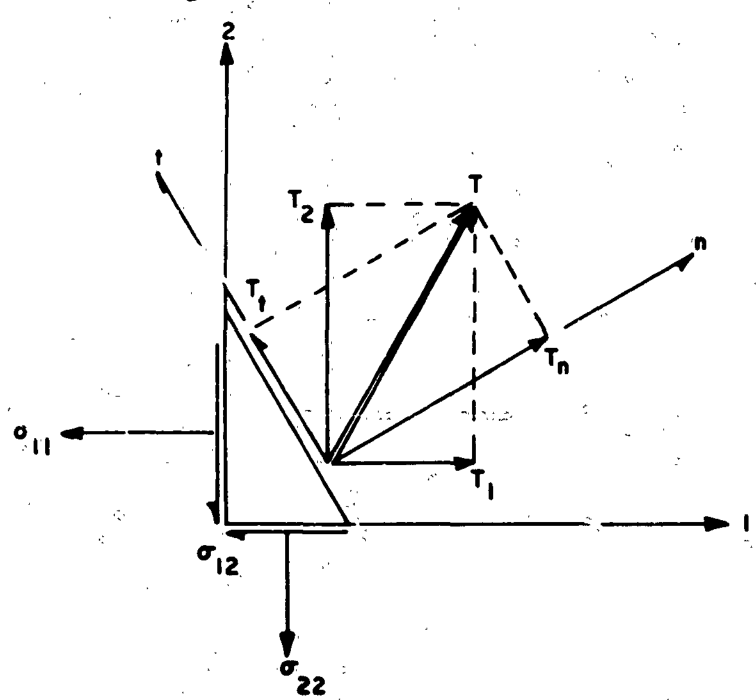


Figure 3. Stress Tensor in 2-Space

The normal component of the surface traction, $T_n$, can be obtained by

$$T_n = T_i n_i = \sigma_{ji} n_j n_i \tag{30}$$

In 2-space

$$T_n = \sigma_{11} n_1^2 + 2\sigma_{12} n_1 n_2 + \sigma_{22} n_2^2 \tag{31}$$

The tangential component of the surface traction, $T_t$, can be obtained as follows: From the directional cosines $n_i$ of the normal to the surface, one can find the directional cosines $t_i$ in the tangential direction by transforming $n_i$ through 90°. Since $n_i$ is a first rank tensor,

$$t_i = t_{ij} n_j \tag{32}$$

where

$$t_{ij} = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{33}$$

Thus

$$t_1 = -n_2 \,,\; t_2 = n_1 \,,\; t_3 = n_3 \tag{34}$$

Hence

$$T_t = T_i t_i = \sigma_{ji} n_j t_i \tag{35}$$

For 2-space

$$\begin{aligned} T_t &= \sigma_{11} n_1 t_1 + \sigma_{12} n_1 t_2 + \sigma_{21} n_2 t_1 + \sigma_{22} n_2 t_2 \\ &= -\sigma_{11} n_1 n_2 + \sigma_{12} n_1^2 - \sigma_{21} n_2^2 + \sigma_{22} n_2 n_1 \end{aligned} \tag{36}$$

Since

$$\begin{aligned} T_1 &= T_n n_1 - T_t n_2 \\ T_2 &= -T_n t_1 + T_t t_2 \end{aligned} \tag{37}$$

From Equations 31 and 36

$$\begin{aligned} T_1 = {} & \sigma_{11} n_1^3 + 2\sigma_{12} n_1^2 n_2 + \sigma_{22} n_1 n_2^2 + \sigma_{11} n_1 n_2^2 + \sigma_{12}(n_2^2 - n_1^2) n_2 \\ & - \sigma_{22} n_1 n_2^2 = \sigma_{11} n_1 + \sigma_{12} n_2 \end{aligned} \tag{38}$$

Similarly,

$$\begin{aligned} T_2 = {} & \sigma_{11} n_1^2 n_2 + 2\sigma_{12} n_1 n_2^2 + \sigma_{22} n_2^3 - \sigma_{11} n_1^2 n_2 - \sigma_{12}(n_2^2 - n_1^2) n_1 \\ & + \sigma_{22} n_1^2 n_2 = \sigma_{12} n_1 + \sigma_{22} n_2 \end{aligned} \tag{39}$$

Equations 38 and 39 agree with Equation 29, as expected. Finally, one can easily show that

$$\begin{aligned} T_n^2 + T_t^2 &= (T_i n_i)^2 + (T_i t_i)^2 \\ &= (T_1 n_1 + T_2 n_2)^2 + (T_1 t_1 + T_2 t_2)^2 \\ &= (T_1 n_1 + T_2 n_2)^2 + (-T_1 n_2 + T_2 n_1)^2 \\ &= T_1^2 + T_2^2 \\ &= T^2 \end{aligned} \tag{40}$$

The equilibrium equation of a continuous medium can be derived from the conservation principle of linear momentum, as follows:

$$\int_A T_i dA + \int_V B_i dV = \int_V \rho \ddot{u}_i dV \tag{41}$$

where $T_i$ = surface traction, $B_i$ = body force, $\rho$ = density, $u_i$ = displacement, double dots = second derivative with respect to time, A = surface area, V = volume enclosed by A.

From Equations 27 and 28

$$\int_A T_i dA = \int_A \sigma_{ji} n_j dA = \int_V \sigma_{ji,j} dV \tag{42}$$

Since the volume is arbitrary,

$$\sigma_{ji,j} + B_i = \rho \ddot{u}_i \tag{43}$$

In the absence of body forces and time effect, the static equilibrium equation is

$$\sigma_{ji,j} = 0 \tag{44}$$

From the conservation principle of angular momentum that if the resultant moment due to body and surface forces vanish and there is no other internal or external sources of moment, it can be shown that the stress tensor must be symmetric, i.e.,

$$\sigma_{ij} = \sigma_{ji} \tag{45}$$

In view of the symmetry property of stress, Equation 44 can be written as

$$\sigma_{ij,j} = 0 \tag{46}$$

This equation in 3-space can be expanded, as follows:

$$\begin{aligned} \frac{\partial \sigma_{11}}{\partial x_1} + \frac{\partial \sigma_{12}}{\partial x_2} + \frac{\partial \sigma_{13}}{\partial x_3} &= 0 \\ \frac{\partial \sigma_{12}}{\partial x_1} + \frac{\partial \sigma_{22}}{\partial x_2} + \frac{\partial \sigma_{23}}{\partial x_3} &= 0 \\ \frac{\partial \sigma_{13}}{\partial x_1} + \frac{\partial \sigma_{23}}{\partial x_2} + \frac{\partial \sigma_{33}}{\partial x_3} &= 0 \end{aligned} \tag{47}$$

STRAIN TENSOR

Strain is a purely geometric relation. It is independent of the constitution of a body as long as it remains continuous during the process of deformation.

Let points P and Q, separated by a distance ds in an undeformed body, displaced to P' and Q' during a deformation process. The relations between these four points are shown in Figure 4.

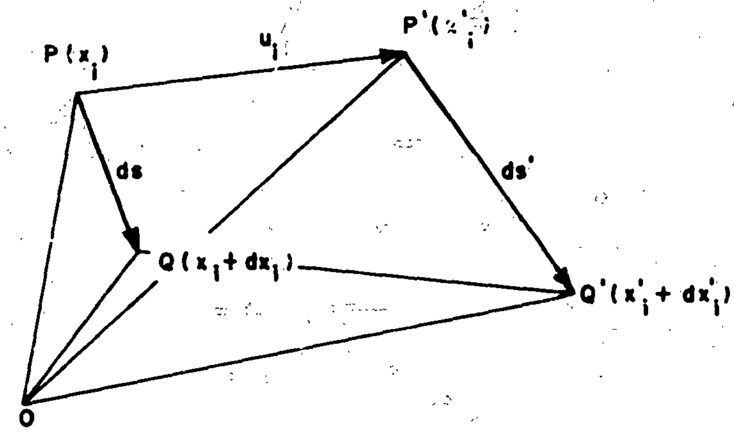


Figure 4. Displacements of Points During Deformation

From Figure 4,

$$u_i = x'_i - x_i \tag{48}$$

$$(ds)^2 = \delta_{ij} dx_i dx_j \tag{49}$$

$$(ds')^2 = \delta_{ij} dx'_i dx'_j \tag{50}$$

If the process of deformation is referred to the final or deformed state (the Eulerian coordinates)

$$x_i = x_i(x'_i) \tag{51}$$

Then

$$dx_i = x_{i,j} dx'_j \tag{52}$$

$$(ds)^2 = x_{m,i} x_{m,j} dx'_i dx'_j \tag{53}$$

$$(ds')^2 = \delta_{ij} dx'_i dx'_j \tag{54}$$

Strain tensor, $e_{ij}$, is defined by

$$(ds')^2 - (ds)^2 = 2e_{ij} dx'_i dx'_j \tag{55}$$

By substituting Equations 53 and 54 into 55, one obtains:

$$2e_{ij} = \delta_{ij} - x_{m,i}x_{m,j} \tag{56}$$

Since

$$x_m = x'_m - u_m$$
$$x_{m,i} = \delta_{mi} - u_{m,i}$$
$$x_{m,j} = \delta_{mj} - u_{m,j}$$

Then

$$\begin{aligned} x_{m,i}x_{m,j} &= (\delta_{mi} - u_{m,i})(\delta_{mj} - u_{m,j}) \\ &= \delta_{ij} - u_{i,j} - u_{j,i} + u_{m,i}u_{m,j} \end{aligned} \tag{57}$$

By using Equation 56

$$2e_{ij} = u_{i,j} + u_{j,i} - u_{m,i}u_{m,j} \tag{58}$$

The Eulerian strain when expanded becomes

$$e_{11} = \frac{\partial u_1}{\partial x'_1} - \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x'_1}\right)^2 + \left(\frac{\partial u_2}{\partial x'_1}\right)^2 + \left(\frac{\partial u_3}{\partial x'_1}\right)^2\right] \tag{59}$$

and

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x'_2} + \frac{\partial u_2}{\partial x'_1}\right) - \frac{1}{2}\left(\frac{\partial u_1}{\partial x'_1}\frac{\partial u_1}{\partial x'_2} + \frac{\partial u_2}{\partial x'_1}\frac{\partial u_2}{\partial x'_2} + \frac{\partial u_3}{\partial x'_1}\frac{\partial u_3}{\partial x'_2}\right) \tag{60}$$

In a similar way, one can derive the Lagrangian strain which refers to the initial or undeformed state, as follows:

$$2e_{ij} = u_{i,j} + u_{j,i} + u_{m,i}u_{m,j} \tag{61}$$

In expanded form

$$e_{11} = \frac{\partial u_1}{\partial x_1} + \frac{1}{2}\left[\left(\frac{\partial u_1}{\partial x_1}\right)^2 + \left(\frac{\partial u_2}{\partial x_1}\right)^2 + \left(\frac{\partial u_3}{\partial x_1}\right)^2\right] \tag{62}$$

and

$$e_{12} = \frac{1}{2}\left(\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right) + \frac{1}{2}\left(\frac{\partial u_1}{\partial x_1}\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\frac{\partial u_2}{\partial x_2} + \frac{\partial u_3}{\partial x_1}\frac{\partial u_3}{\partial x_2}\right) \quad (63)$$

The physical significance of the strain components can be illustrated by two examples:

1) Normal strain.

Let

$$\begin{aligned} ds' &= dx'_1 \text{, implying } dx'_2 = dx'_3 = 0 \\ ds &= (1-E_1)\,ds' = (1-E_1)\,dx'_1 \end{aligned} \quad (64)$$

where

$$E_1 = \frac{ds' - ds}{ds'} \quad (65)$$

From Equation 55,

$$(dx'_1)^2 - (1-E_1)^2 (dx'_1)^2 = 2e_{11}\,dx'^2_1 \quad (66)$$

therefore

$$e_{11} = E_1 - E_1^2/2 \quad (67)$$

Thus, for infinitesimal displacement,

$$e_{11} = E_1 \quad (68)$$

which means that, in the case of infinitesimal strain, the normal component of strain corresponds to axial elongation. For finite strain, Equation 67 will apply.

2) Shear Strain

Let

$$ds' = dx'_2 \text{, with } dx'_1 = dx'_3 = 0 \quad (69)$$

$$d\bar{s}' = d\bar{x}'_3 \text{, with } d\bar{x}'_1 = d\bar{x}'_2 = 0 \quad (70)$$

The angle $\theta$ between the two vectors is

$$\cos\theta = dx_i d\bar{x}_i / ds d\bar{s}$$
$$= x_{i,2} x_{i,3} dx'_2 d\bar{x}'_3 / ds\ d\bar{s}$$

From Equation 56

$$\cos\theta = (\cancel{\delta_{23}} - 2e_{23}) dx'_2 d\bar{x}'_3 / ds d\bar{s} \tag{71}$$

Let

$$\phi = \theta - 90°$$

and it represents the change in the angle between the two elements which was $\theta$ in the initial state and became perpendicular in the final state.

From Equation 71

$$\sin\phi = 2e_{23} dx'_2 d\bar{x}'_3 / ds\ d\bar{s} \tag{72}$$

From Equation 65

$$\sin\phi = 2e_{23} / (1 - E_2)(1 - E_3) \tag{73}$$

From Equation 66

$$\sin\phi = 2e_{23} / \sqrt{(1 - 2e_{22})(1 - 2e_{33})} \tag{74}$$

For infinitesimal strain,

$$\phi = 2e_{23} \tag{75}$$

Thus, the infinitesimal shear strain corresponds to one half of the change in angle from 90°.

The strain-displacement equations for both Eulerian and Lagrangian strains reduce to the same equation in case of infinitesimal strain as follows:

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i}) \tag{76}$$

This strain is a second-rank tensor. The conventional engineering strain is different from the tensorial strain in the shear strain components. The engineering shear strain is related to the total change in an angle, rather than one half of the angle according to Equation 75. For this reason, engineering shear strain is twice the tensorial shear strain.

# SECTION II

# TRANSFORMATION EQUATIONS

Tensors are defined by the following transformation equation:

$$C'_{ij\ldots kl} = t_{im}t_{jn}\ldots t_{ko}t_{lp}C_{mn\ldots op} \tag{77}$$

where $C'_{ij\cdots kl}$ is the transformed $C_{mn\cdots op}$; $t_{ij}$, the transformation matrix. In first, second, and fourth-rank tensors, Equation 77 can be specialized as follows:

$$C'_i = t_{im}C_m \tag{78}$$

$$C'_{ij} = t_{im}t_{jn}C_{mn} \tag{79}$$

$$C'_{ijkl} = t_{im}t_{jn}t_{ko}t_{lp}C_{mnop} \tag{80}$$

## FIRST-RANK TENSORS

Equation 78 represents the following simultaneous equations in 2-space and 3-space:

$$\begin{bmatrix} C'_1 \\ C'_2 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} \\ t_{21} & t_{22} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{81}$$

$$\begin{bmatrix} C'_1 \\ C'_2 \\ C'_3 \end{bmatrix} = \begin{bmatrix} t_{11} & t_{12} & t_{13} \\ t_{21} & t_{22} & t_{23} \\ t_{31} & t_{32} & t_{33} \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \tag{82}$$

Assuming that the transformation is a positive rotation $T^+$, which has the components shown in Equation 11, Equations 81 and 82 become

$$\begin{bmatrix} C'_1 \\ C'_2 \end{bmatrix} = \begin{bmatrix} m & n \\ -n & m \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \end{bmatrix} \tag{83}$$

$$\begin{bmatrix} C'_1 \\ C'_2 \\ C'_3 \end{bmatrix} \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C_1 \\ C_2 \\ C_3 \end{bmatrix} \tag{84}$$

These are the transformation equations of $T^+$ for the first-rank tensors (vectors).

SECOND-RANK TENSORS

For second-rank tensors, the transformation Equation 79 in 2-space can be expanded as follows:

1) When $i = 1$, $j = 1$,

$$C'_{11} = t_{1m} t_{1n} C_{mn}$$

Summing m gives

$$C'_{11} = t_{11} t_{1n} C_{1n} + t_{12} t_{1n} C_{2n} \tag{85}$$

Summing n gives

$$C'_{11} = t_{11}(t_{11} C_{11} + t_{12} C_{12}) + t_{12}(t_{11} C_{21} + t_{12} C_{22})$$

2) When $i = 2$, $j = 2$,

$$C'_{22} = t_{2m} t_{2n} C_{mn}$$

Summing m gives

$$C'_{22} = t_{21} t_{2n} C_{1n} + t_{22} t_{2n} C_{2n} \tag{86}$$

Summing n gives

$$C'_{22} = t_{21}(t_{21} C_{11} + t_{22} C_{12}) + t_{22}(t_{21} C_{21} + t_{22} C_{22})$$

3) When $i = 1$, $j = 2$,

$$C'_{12} = t_{1m} t_{2n} C_{mn}$$

Summing m gives

$$C'_{12} = t_{11} t_{2n} C_{1n} + t_{12} t_{2n} C_{2n} \tag{87}$$

Summing n gives

$$C'_{12} = t_{11}(t_{21} C_{11} + t_{22} C_{12}) + t_{12}(t_{21} C_{21} + t_{22} C_{22})$$

4) When $i = 2, j = 1$,

$$C'_{21} = t_{2m} t_{1n} C_{mn}$$

Summing m gives

$$C'_{21} = t_{21} t_{1n} C_{1n} + t_{22} t_{1n} C_{2n} \tag{88}$$

Summing n gives

$$C'_{21} = t_{21} (t_{11} C_{11} + t_{12} C_{12}) + t_{22} (t_{11} C_{21} + t_{12} C_{22})$$

From Equation 11, the components of T-matrix in 2-space are

$$T = \begin{pmatrix} m & n \\ -n & m \end{pmatrix} \tag{89}$$

By substituting $t_{ij}$ into Equations 85 through 88, one obtains

$$\begin{aligned} C'_{11} &= m^2 C_{11} + n^2 C_{22} + mnC_{12} + mnC_{21} \\ C'_{22} &= n^2 C_{11} + m^2 C_{22} - mnC_{12} - mnC_{21} \\ C'_{12} &= -mnC_{11} + mnC_{22} + m^2 C_{12} - n^2 C_{21} \\ C'_{21} &= -mnC_{11} + mnC_{22} - n^2 C_{12} + m^2 C_{21} \end{aligned} \tag{90}$$

In a matrix form Equation 90 becomes

$$\begin{bmatrix} C'_{11} \\ C'_{22} \\ C'_{12} \\ C'_{21} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & mn & mn \\ n^2 & m^2 & -mn & -mn \\ -mn & mn & m^2 & -n^2 \\ -mn & mn & -n^2 & m^2 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{22} \\ C_{12} \\ C_{21} \end{bmatrix} \tag{91}$$

If $C_{ij}$ is a symmetric tensor,

$$C_{ij} = C_{ji} \tag{92}$$

therefore

$$C_{12} = C_{21}$$

Equation 91 can be simplified as follows:

$$\begin{bmatrix} C'_{11} \\ C'_{22} \\ C'_{12} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 2mn \\ n^2 & m^2 & -2mn \\ -mn & mn & m^2-n^2 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{22} \\ C_{12} \end{bmatrix} \tag{93}$$

These are the transformation equations of $T^+$ for second-rank tensors which include the stress and inertial tensors.

In 3-space, the range for the indices will be 1, 2, and 3. If the rotation remains the same as $T^+$ before, the following results will be obtained:

1) When $i = j = 1$

$$\begin{aligned} C'_{11} &= t_{1m} t_{1n} C_{mn} \\ &= t_{11} t_{1n} C_{1n} + t_{12} t_{1n} C_{2n} + t_{13} t_{1n} C_{3n} \\ &= t_{11}(t_{11} C_{11} + t_{12} C_{12} + t_{13} C_{13}) \\ &\quad + t_{12}(t_{11} C_{21} + t_{12} C_{22} + t_{13} C_{23}) \\ &\quad + t_{13}(t_{11} C_{31} + t_{12} C_{32} + t_{13} C_{33}) \end{aligned} \tag{94}$$

Since for the present T-matrix,

$$t_{13} = t_{23} = t_{31} = t_{32} = 0 \tag{95}$$

then

$$C'_{11} = m^2 C_{11} + n^2 C_{22} + 2mn C_{12} \tag{96}$$

where $C_{ij}$ is assumed to be a symmetric tensor, and the results of Equation 96 are the same as that in Equation 93.

2) When i = 2, j = 2,

$$
\begin{aligned}
C'_{22} &= t_{2m} t_{2n} C_{mn} \\
&= t_{21} t_{2n} C_{1n} + t_{22} t_{2n} C_{2n} + t_{23} t_{2n} C_{3n} \qquad (97)\\
&= n^2 C_{11} + m^2 C_{22} - 2mnC_{12}
\end{aligned}
$$

This is the same as Equation 93 because of the relations in Equation 95.

3) When i = 3, j = 3,

$$
\begin{aligned}
C'_{33} &= t_{3m} t_{3n} C_{mn} \\
&= t_{31} t_{3n} C_{1n} + t_{32} t_{3n} C_{2n} + t_{33} t_{3n} C_{3n} \qquad (98)\\
&= t_{33}^2 C_{33} = C_{33}
\end{aligned}
$$

where Equation 95 is used and $t_{33} = 1$.

4) When i = 2, j = 3,

$$
\begin{aligned}
C'_{23} &= t_{2m} t_{3n} C_{mn} \\
&= t_{21} t_{3n} C_{1n} + t_{22} t_{3n} C_{2n} + t_{23} t_{3n} C_{3n} \qquad (99)\\
&= t_{21} t_{33} C_{13} + t_{22} t_{33} C_{23} \\
&= -nC_{13} + mC_{23}
\end{aligned}
$$

5) When i = 3, j = 1,

$$
\begin{aligned}
C'_{31} &= t_{3m} t_{1n} C_{mn} \\
&= t_{31} t_{1n} C_{1n} + t_{32} t_{1n} C_{2n} + t_{33} t_{1n} C_{3n} \qquad (100)\\
&= t_{33} (t_{11} C_{31} + t_{12} C_{32}) \\
&= mC_{31} + nC_{32}
\end{aligned}
$$

6) When $i = 1$, $j = 2$,

$$\begin{aligned} C'_{12} &= t_{1m} t_{2n} C_{mn} \\ &= t_{11} t_{2n} C_{1n} + t_{12} t_{2n} C_{2n} + t_{13} t_{2n} C_{3n} \\ &= t_{11} (t_{21} C_{11} + t_{22} C_{12}) + t_{12} (t_{21} C_{21} + t_{22} C_{22}) \\ &= -mnC_{11} + mnC_{22} + (m^2 - n^2) C_{12} \end{aligned} \tag{101}$$

This is the same as Equation 93. Thus, the transformation equations for 3-space, which are equivalent to Equation 93 in 2-space, are

$$\begin{bmatrix} C'_{11} \\ C'_{22} \\ C'_{33} \\ C'_{23} \\ C'_{31} \\ C'_{12} \end{bmatrix} = \begin{bmatrix} m^2 & n^2 & 0 & 0 & 0 & 2mn \\ n^2 & m^2 & 0 & 0 & 0 & -2mn \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & m & -n & 0 \\ 0 & 0 & 0 & n & m & 0 \\ -mn & mn & 0 & 0 & 0 & m^2-n^2 \end{bmatrix} \begin{bmatrix} C_{11} \\ C_{22} \\ C_{33} \\ C_{23} \\ C_{31} \\ C_{12} \end{bmatrix} \tag{102}$$

Note that $C_{33}$ component is invariant, i.e., $C'_{33} = C_{33}$, and $C_{32}$ and $C_{31}$ interact with each other and are not coupled with the remaining components. If $C_{ij}$ represents a stress tensor,

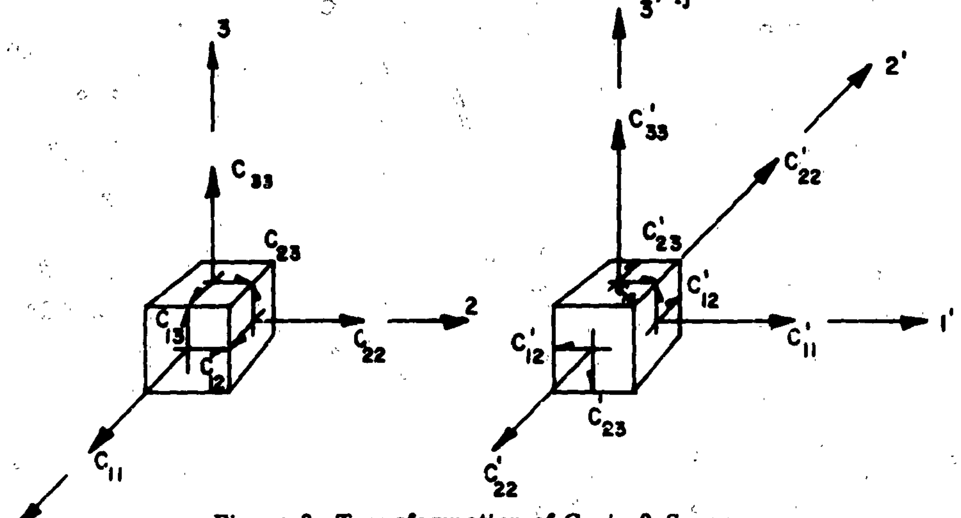


Figure 3. Transformation of $C_{ij}$ in 3-Space

$C_{23}$ and $C_{31}$ would be the transverse shear stresses of a plate with its plane parallel to the 1-2 plane. The states of stress before and after a transformation of $\theta = 90°$ are shown in Figure 3.

Note that when $\theta = 90°$, $m = 0$, $n = 1$. These results, when substituted into Equation 102 yield

$$\begin{aligned} C'_{11} &= C_{22} \\ C'_{22} &= C_{11} \\ C'_{33} &= C_{33} \\ C'_{23} &= -C_{31} \\ C'_{31} &= C_{23} \\ C'_{12} &= -C_{12} \end{aligned} \tag{103}$$

The relations in Equation 103 agree with the results shown in Figure 3. The shear components shown in this figure represent positive quantities (the shear diagonals lie between the positive directions of the coordinate axes).

CONTRACTED NOTATION

A further simplification of the indicial notation is possible with the contracted notation. In dealing with fourth-rank tensors, the contracted notation reduces the number of free indices from 4 to 2 but expands the range from 3 to 9. The number of components, according to Equation 1, remains at $3^4 = 81$ and $9^2 = 81$ for the normal and contracted indicial notation, respectively. But if symmetry properties are introduced, the contracted notation can be used to obtain a considerable amount of simplification.

The fourth-rank tensor of interest now is the elastic stiffness or compliance matrices, $S_{ijkl}$ or $C_{ijkl}$. They appear in the generalized Hooke's laws as follows:

$$\sigma_{ij} = C_{ijkl} e_{kl} \tag{104}$$

$$e_{ij} = S_{ijkl} \sigma_{kl} \tag{105}$$

where $\sigma_{ij}$ = stress tensor, $e_{ij}$ = strain tensor, and both tensors have the following 9 components in 3-space:

$$\begin{matrix} \sigma_{11} & e_{11} \\ \sigma_{22} & e_{22} \\ \sigma_{33} & e_{33} \\ \sigma_{23} & e_{23} \\ \sigma_{31} & e_{31} \\ \sigma_{12} & e_{12} \\ \sigma_{32} & e_{32} \\ \sigma_{13} & e_{13} \\ \sigma_{21} & e_{21} \end{matrix} \tag{106}$$

The contracted notation is established by arbitrarily replacing the double- index system for a second-rank tensor with a single-index system. Each component in Equation 106 can be represented by a single-index system from 1 to 9; e.g., as shown in Table II.

TABLE II

CONVERSION BETWEEN NORMAL AND CONTRACTED NOTATIONS

| Normal Notation | | Contracted Notation | |
|---|---|---|---|
| $\sigma_{11}$ | $e_{11}$ | $\sigma_1$ | $e_1$ |
| $\sigma_{22}$ | $e_{22}$ | $\sigma_2$ | $e_2$ |
| $\sigma_{33}$ | $e_{33}$ | $\sigma_3$ | $e_3$ |
| $\sigma_{23}$ | $2e_{23}$ | $\sigma_4$ | $e_4$ |
| $\sigma_{31}$ | $2e_{31}$ | $\sigma_5$ | $e_5$ |
| $\sigma_{12}$ | $2e_{12}$ | $\sigma_6$ | $e_6$ |
| $\sigma_{32}$ | $2e_{32}$ | $\sigma_7$ | $e_7$ |
| $\sigma_{13}$ | $2e_{13}$ | $\sigma_8$ | $e_8$ |
| $\sigma_{21}$ | $2e_{21}$ | $\sigma_9$ | $e_9$ |

In contracted notation, engineering strain is used instead of tensorial strain and Equations 104 and 105 can be written as:

$$e_i = S_{ij}\sigma_j \tag{107}$$

$$\sigma_i = C_{ij}e_j \tag{108}$$

where $i, j = 1, 2, \ldots,$ and 9. In this notation, the range and summation conventions are retained. But some modifications in the interpretation of the indices must be made. First, the range of free indices no longer corresponds to the number of dimensions in space. Secondly, the tensorial rank no longer corresponds to the number of free indices. Finally, the contracted notation is an artificial notation which happens to provide significant simplification in the representation of the generalized Hooke's laws but only second and fourth rank tensors where 3-space is involved. The use of contracted notation for other tensorial relations should be limited to special cases.

Returning to Equations 107 and 108, the range of the indices can be reduced from 9 to 6 if the stress and strain tensors are symmetric i.e.,

$$\sigma_{ij} = \sigma_{ji}, \text{ and } e_{ij} = e_{ji} \tag{109}$$

which, in contracted notation, means

$$\sigma_7 = \sigma_4,\quad \sigma_8 = \sigma_5,\quad \sigma_9 = \sigma_6 \tag{110}$$

$$e_7 = e_4,\quad e_8 = e_5,\quad e_9 = e_6 \tag{111}$$

Thus, the symmetries shown in Equation 109 reduce the number of components of $S_{ij}$ and $C_{ij}$ from 81 to 36 in 3-space, and from 16 to 9 in 2-space. Equation 6 in contracted notation becomes:

$$a_{ij} = \begin{bmatrix} a_{11} & a_{12} & a_{16} & a_{19} \\ a_{21} & a_{22} & a_{26} & a_{29} \\ a_{61} & a_{62} & a_{66} & a_{69} \\ a_{91} & a_{92} & a_{96} & a_{99} \end{bmatrix} \tag{112}$$

where the replacement of the 4-index system to a 2-index system follows the relationship in Table II. If the $a_{ij}$ is a compliance matrix in the generalized Hooke's law and the stress and strain tensors are symmetric. Equations 110 and 111 can be used to simplify Equation 112 as follows:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{16} \\ S_{21} & S_{22} & S_{26} \\ S_{61} & S_{62} & S_{66} \end{bmatrix} \tag{113}$$

The $S_{ij}$ in 3-space will be:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16} \\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26} \\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36} \\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46} \\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56} \\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66} \end{bmatrix} \tag{114}$$

An additional symmetry property can be established from the strain energy consideration. If the existence of an elastic potential is assumed then an increment work per unit volume is,

$$dW = \sigma_i de_i \tag{115}$$

then substitute

$$\sigma_i = C_{ij} e_j \tag{116}$$

$$dW = C_{ij} e_j de_i \tag{117}$$

By integrating, one gets

$$W = \frac{1}{2} C_{ij} e_i e_j \tag{118}$$

Similarly, one can show that

$$W = \frac{1}{2} S_{ij} \sigma_i \sigma_j \tag{119}$$

From the elastic potentials, one can derive the generalized Hooke's law, as follows:

$$\frac{\partial W}{\partial e_i} = \sigma_i = C_{ij} e_j \tag{120}$$

and

$$\frac{\partial^2 W}{\partial e_i \partial e_j} = C_{ij} \tag{121}$$

Similarly,

$$\frac{\partial^2 W}{\partial e_j \partial e_i} = C_{ji} \tag{122}$$

Since the order of differentiation is immaterial, then

$$C_{ij} = C_{ji} \tag{123}$$

i.e., the stiffness matrix must be symmetric with respect to the indices in the contracted notation. In a similar manner, one can show that

$$S_{ij} = S_{ji} \tag{124}$$

This additional symmetry will simplify Equations 113 and 114, such that there are only 6 and 21 independent components in 2-space and 3-space, respectively.

The conversion between the normal and contracted notations for $S_{ij}$ and $C_{ij}$ cannot be derived directly from Table II. Some correction factors must be introduced because of the difference between the engineering and tensorial strains. The factors can be established by

expanding Equations 104, 105, 107, and 108, and by comparing them term by term. For example, for $i = j = 1$, from Equation 104,

$$\begin{aligned} e_{11} = {} & S_{1111}\sigma_{11} + S_{1122}\sigma_{22} + S_{1133}\sigma_{33} \\ & + (S_{1123} + S_{1132})\sigma_{23} + (S_{1131} + S_{1113})\sigma_{31} \\ & + (S_{1112} + S_{1121})\sigma_{12} \end{aligned} \tag{125}$$

From Equation 107

$$e_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3 + S_{14}\sigma_4 + S_{15}\sigma_5 + S_{16}\sigma_6 \tag{126}$$

Comparison of Equations 125 and 126, and an assumption that the matrix is symmetrical yields

$$\begin{aligned} S_{1111} &= S_{11} \\ S_{1122} &= S_{12} \\ S_{1133} &= S_{13} \\ 2S_{1123} &= S_{14} \\ 2S_{1131} &= S_{15} \\ 2S_{1112} &= S_{16} \end{aligned} \tag{127}$$

By repeating the process similar to that shown in Equations 125 and 126, one can establish the following conversions of the components of the compliance matrix:

$$\begin{aligned} S_{ijkl} &= S_{qr} \text{ for } q, r = 1, 2, \text{or } 3 \\ 2S_{ijkl} &= S_{qr} \text{ for } q = 1, 2, \text{ or } 3;\ r = 4, 5, \text{ or } 6 \\ & \qquad\quad \text{or } q = 4, 5, \text{ or } 6;\ r = 1, 2, \text{ or } 3 \\ 4S_{ijkl} &= S_{qr} \text{ for } q, r = 4, 5, \text{or } 6 \end{aligned} \tag{128}$$

The conversion factors for the stiffness matrix $C_{ij}$ can be similarly established. In $i = j = 1$,

$$\begin{aligned} \sigma_{11} = {} & C_{1111}e_{11} + C_{1122}e_{22} + C_{1133}e_{33} \\ & + (C_{1123} + C_{1132})e_{23} + (C_{1131} + C_{1113})e_{31} \\ & + (C_{1112} + C_{1121})e_{12} \end{aligned} \tag{129}$$

From Equation 56

$$\sigma_1 = C_{11}e_1 + C_{12}e_2 + C_{13}e_3 + C_{14}e_4 + C_{15}e_5 + C_{16}e_6 \tag{130}$$

From Table II,

$$2e_{23} = e_4 , \quad 2e_{31} = e_5 , \quad 2e_{12} = e_6 \tag{131}$$

From the last three numbered equations,

$$\begin{aligned} C_{1111} &= C_{11} , \quad C_{1122} = C_{12} , \quad C_{1133} = C_{13} , \\ C_{1123} &= C_{14} , \quad C_{1131} = C_{15} , \quad C_{1112} = C_{16} . \end{aligned} \tag{132}$$

By repeating the process for the other relations in the generalized Hooke's law, one can establish that

$$C_{ijkl} = C_{qr} \tag{133}$$

Thus, the conversion factor is unity for all components of the stiffness matrix. But for the compliance matrix, relations in Equations 128 must be followed. The contracted notation must be handled with care. The tranformation equations must be derived using the normal indicial notation. The relation between the components of the compliance and stiffness matrix of the two notations must include the proper conversion factors as shown in Equations 128 and 133.

The use of the contracted notation has often been inconsistent in many current publications and reports. In many cases, tensorial strain is retained. In other cases, both single and double index systems are used simultaneously, e.g., $e_1, e_2$, and $e_{12}$ for the strain components in 2-space. Finally, $e_{12}$ is sometimes represented by $e_3$ instead of $e_6$.

The conversion between the normal and contracted notations as listed in Table II and Equations 110 and 111 are recommended because a consistent notation between 3-space and 2-space, and consistent operations (range and summation conventions) can be retained. In the contracted notation, fourth-rank tensors are represented by double-index quantities ($C_{ij}$, $S_{ij}$); second-rank tensors, by single-index quantities. The range for 3-space is 1, 2, 3, 4, 5, 6, and 2-space range is 1, 2, 6. The latter range is probably better than 1, 2, 3 because it avoids the similarity between the range of 3-space of the normal notation and the range of 2-space of contracted notations.

Finally, the contracted notation as listed in Table II may be considered authoritative because it follows the notation listed in many textbooks (References 1, 2, and 3).

## FOURTH-RANK TENSORS

The transformation equations for fourth-rank tensors contain 81 equations. With the aid of symmetry properties, e.g., $\sigma_{ij} = \sigma_{ji}, e_{ij} = e_{ji}$, and $S_{ij} = S_{ji}$ and $C_{ij} = C_{ji}$ (the last two equations are in contracted notation), there are only 21 equations. A further simplification can be introduced if the T-matrix is limited to a rotation about the 3-axis, as shown in Figure 1, where,

$$t_{ij} = \begin{bmatrix} m & n & 0 \\ -n & m & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{134}$$

The transformation equations must be derived from the normal indicial notation, not the contracted notation, because in the latter notation artificial simplifications are introduced. The number of free indices no longer corresponds to the tensorial rank. Repeating the form of Equation 80, in which

$$C'_{ijkl} = t_{im}t_{jn}t_{ko}t_{lp}C_{mnop}$$

the transformation of $S_{ijkl}$ is, similarly

$$S'_{ijkl} = t_{im}t_{jn}t_{ko}t_{lp}S_{mnop} \tag{135}$$

Further, in 3-space, by taking advantage of $t_{13} = t_{23} = t_{31} = t_{32} = 0$:

$$S'_{1111} = t_{1m}t_{1n}t_{1o}t_{1p}S_{mnop}$$

Summing m gives

$$S'_{1111} = t_{11}t_{1n}t_{1o}t_{1p}S_{1nop} + t_{12}t_{1n}t_{1o}t_{1p}S_{2nop} + t_{13}t_{1n}t_{1o}t_{1p}S_{3nop}$$

Summing n gives

$$\begin{aligned} S'_{1111} = {} & t_{11}(t_{11}t_{1o}t_{1p}S_{11op} + t_{12}t_{1o}t_{1p}S_{12op} + \cancel{t_{13}}t_{1o}t_{1p}S_{13op}) \\ & + t_{12}(t_{11}t_{1o}t_{1p}S_{21op} + t_{12}t_{1o}t_{1p}S_{22op} + \cancel{t_{13}}t_{1o}t_{1p}S_{23op}) \end{aligned}$$

Summing o gives

$$\begin{aligned} S'_{1111} = {} & t_{11}\big[t_{11}(t_{11}t_{1p}S_{111p} + t_{12}t_{1p}S_{112p} + \cancel{t_{13}}t_{1p}S_{113p}) \\ & \quad + t_{12}(t_{11}t_{1p}S_{121p} + t_{12}t_{1p}S_{122p} + \cancel{t_{13}}t_{1p}S_{123p})\big] \\ & + t_{12}\big[t_{11}(t_{11}t_{1p}S_{211p} + t_{12}t_{1p}S_{212p} + \cancel{t_{13}}t_{1p}S_{213p}) \\ & \quad + t_{12}(t_{11}t_{1p}S_{221p} + t_{12}t_{1p}S_{222p} + \cancel{t_{13}}t_{1p}S_{223p})\big] \end{aligned}$$

Summing p gives

$$\begin{aligned}
S'_{1111} = \; & t_{11}\Big\{t_{11}\Big[t_{11}(t_{11}S_{1111} + t_{12}S_{1112} + \cancel{t_{13}}S_{1113}) \\
& \qquad + t_{12}(t_{11}S_{1121} + t_{12}S_{1122} + \cancel{t_{13}}S_{1123})\Big] \\
& \quad + t_{12}\Big[t_{11}(t_{11}S_{1211} + t_{12}S_{1212} + \cancel{t_{13}}S_{1213}) \\
& \qquad + t_{12}(t_{11}S_{1221} + t_{12}S_{1222} + \cancel{t_{13}}S_{1223})\Big]\Big\} \\
& + t_{12}\Big\{t_{11}\Big[t_{11}(t_{11}S_{2111} + t_{12}S_{2112} + \cancel{t_{13}}S_{2113}) \\
& \qquad + t_{12}(t_{11}S_{2121} + t_{12}S_{2122} + \cancel{t_{13}}S_{2123})\Big] \\
& \quad + t_{12}\Big[t_{11}(t_{11}S_{2211} + t_{12}S_{2212} + \cancel{t_{13}}S_{2213}) \\
& \qquad + t_{12}(t_{11}S_{2221} + t_{12}S_{2222} + \cancel{t_{13}}S_{2223})\Big]\Big\}
\end{aligned}$$

$$\begin{aligned}
= \; & m^4 S_{1111} + m^3 n S_{1112} + m^3 n S_{1121} + m^2 n^2 S_{1122} \\
& + m^3 n S_{1211} + m^2 n^2 S_{1212} + m^2 n^2 S_{1221} + m n^3 S_{1222} \\
& + m^3 n S_{2111} + m^2 n^2 S_{2112} + m^2 n^2 S_{2121} + m n^3 S_{2122} \\
& + m^2 n^2 S_{2211} + m n^3 S_{2212} + m n^3 S_{2221} + n^4 S_{2222}
\end{aligned} \tag{136}$$

Contracted notation, with the proper conversion factors, modifies Equation 136 to

$$\begin{aligned}
S'_{11} = \; & m^4 S_{11} + m^3 n S_{16}/2 + m^3 n S_{16}/2 + m^2 n^2 S_{12} \\
& + m^3 n S_{61}/2 + m^2 n^2 S_{66}/4 + m^2 n^2 S_{66}/4 + m n^3 S_{62}/2 \\
& + m^3 n S_{61}/2 + m^2 n^2 S_{66}/4 + m^2 n^2 S_{66}/4 + m n^3 S_{62}/2 \\
& + m^2 n^2 S_{21} + m n^3 S_{26}/2 + m n^3 S_{26}/2 + n^4 S_{22}
\end{aligned} \tag{137}$$

From Equation 124, where $S_{ij}$ is symmetric, Equation 137 can be further reduced to

$$S'_{11} = m^4 S_{11} + 2m^2 n^2 S_{12} + 2m^3 n S_{16} + n^4 S_{22} + 2mn^3 S_{26} + m^2 n^2 S_{66} \tag{138}$$

By following the steps described above, the transformation equations for the remaining 20 components of the compliance matrix and all the components of the stiffness matrix can be derived. The transformation, like Equation 138, applies to a transformation consisting of a proper rotation about the 3-axis. This is a very special transformation. A general transformation, for which all components of the T-matrix are nonzero will result in transformation equations considerably more complicated than Equation 138, in which only 6 out of a total of 21 components affect the $S'_{11}$ component. The transformation equations for $S_{ij}$ and $C_{ij}$, when

**subjected to a proper rotation about the 3-axis, can best be presented in the following tabular forms. The use of the tables can be seen by comparing Equation 138 with the first row of the first table. In case of the stiffness matrix $C_{ij}$, appropriate factors shown in the column and row headings must be properly incorporated, as follows:**

$$C'_{11} = m^4 C_{11} + 2m^2n^2 C_{12} + 4m^3 n C_{16} + n^4 C_{22} + 4mn^3 C_{26} + 4m^2n^2 C_{66} \tag{139}$$

(140)

| | $S_{11}(C_{11})$ | $S_{12}(C_{12})$ | $S_{16}(2C_{16})$ | $S_{22}(C_{22})$ | $S_{26}(2C_{26})$ | $S_{66}(4C_{66})$ |
|---|---|---|---|---|---|---|
| $S'_{11}(C'_{11})$ | $m^4$ | $2m^2n^2$ | $2m^3n$ | $n^4$ | $2mn^3$ | $m^2n^2$ |
| $S'_{12}(C'_{12})$ | $m^2n^2$ | $m^4+n^4$ | $mn^3 - m^3n$ | $m^2n^2$ | $m^3n - mn^3$ | $-m^2n^2$ |
| $S'_{16}(2C'_{16})$ | $-2m^3n$ | $2m^3n - 2mn^3$ | $m^4 - 3m^2n^2$ | $2mn^3$ | $3m^2n^2 - n^4$ | $m^3n - mn^3$ |
| $S'_{22}(C'_{22})$ | $n^4$ | $2m^2n^2$ | $-2mn^3$ | $m^4$ | $-2m^3n$ | $m^2n^2$ |
| $S'_{26}(2C'_{26})$ | $-2mn^3$ | $2mn^3 - 2m^3n$ | $3m^2n^2 - n^4$ | $2m^3n$ | $m^4 - 3m^2n^2$ | $mn^3 - m^3n$ |
| $S'_{66}(4C'_{66})$ | $4m^2n^2$ | $-8m^2n^2$ | $4mn^3 - 4m^3n$ | $4m^2n^2$ | $4m^3n - 4mn^3$ | $(m^2 - n^2)^2$ |

(141)

| | $S_{13}(C_{13})$ | $S_{23}(C_{23})$ | $S_{36}(2C_{36})$ |
|---|---|---|---|
| $S'_{13}(C'_{13})$ | $m^2$ | $n^2$ | $mn$ |
| $S'_{23}(C'_{23})$ | $n^2$ | $m^2$ | $-mn$ |
| $S'_{36}(2C'_{36})$ | $-2mn$ | $2mn$ | $m^2 - n^2$ |

(142)

| | $S_{44}(C_{44})$ | $S_{45}(C_{45})$ | $S_{55}(C_{55})$ |
|---|---|---|---|
| $S'_{44}(C'_{44})$ | $m^2$ | $-2mn$ | $n^2$ |
| $S'_{45}(C'_{45})$ | $mn$ | $m^2 - n^2$ | $-mn$ |
| $S'_{55}(C'_{55})$ | $n^2$ | $2mn$ | $m^2$ |

| | $S_{34}(C_{34})$ | $S_{35}(C_{35})$ |
|---|---|---|
| $S'_{34}(C'_{34})$ | $m$ | $-n$ |
| $S'_{35}(C'_{35})$ | $n$ | $m$ |

(143)

| | $S_{14}(C_{14})$ | $S_{15}(C_{15})$ | $S_{24}(C_{24})$ | $S_{25}(C_{25})$ | $S_{46}(2C_{46})$ | $S_{56}(2C_{56})$ |
|---|---|---|---|---|---|---|
| $S'_{14}(C'_{14})$ | $m^3$ | $-m^2n$ | $mn^2$ | $-n^3$ | $m^2n$ | $-mn^2$ |
| $S'_{15}(C'_{15})$ | $m^2n$ | $m^3$ | $n^3$ | $mn^2$ | $mn^2$ | $m^2n$ |
| $S'_{24}(C'_{24})$ | $mn^2$ | $-n^3$ | $m^3$ | $-m^2n$ | $-m^2n$ | $mn^2$ |
| $S'_{25}(C'_{25})$ | $n^3$ | $mn^2$ | $m^2n$ | $m^3$ | $-mn^2$ | $-m^2n$ |
| $S'_{46}(2C'_{46})$ | $-2m^2n$ | $2mn^2$ | $2m^2n$ | $-2mn^2$ | $m^3 - mn^2$ | $n^3 - m^2n$ |
| $S'_{56}(2C'_{56})$ | $-2mn^2$ | $-2m^2n$ | $2mn^2$ | $2m^2n$ | $m^2n - n^3$ | $m^3 - mn^2$ |

(144)

$$S'_{33}(C'_{33}) = S_{33}(C_{33}) \tag{145}$$

Note that under a particular transformation, which in this case is a rotation about the 3-axis, the components of $S_{ij}$ and $C_{ij}$ are arranged in 6 groups, each of which is numbered as an equation between Equations 140 and 145. The components of each group will interact with one another, but are completely uncoupled from the other groups. This information is useful in the study of elastic symmetry. If for a given material all the components within a group are zero, they will remain zero for all angles of rotation about the 3-axis.

## INVARIANTS

There are a number of invariants associated with the compliance and stiffness matrices with respect to the rotation about the 3-axis:

$$I_1 = S'_{11} + S'_{22} + 2S'_{12}$$

From Equation 140

$$\begin{aligned} I_1 = \; & (m^4 + n^4 + 2m^2n^2)S_{11} + (2m^2n^2 + 2m^2n^2 + 2m^4 + 2n^4)S_{12} \\ & + (2m^3n - 2mn^3 + 2mn^3 - 2m^3n)S_{16} + (n^4 + m^4 + 2m^2n^2)S_{22} \\ & + (2mn^3 - 2m^3n + 2m^3n - 2mn^3)S_{26} + (m^2n^2 + m^2n^2 - 2m^2n^2)S_{66} \\ = \; & S_{11} + S_{22} + 2S_{12} \end{aligned} \tag{146}$$

$$\begin{aligned}
I_2 &= S'_{66} - 4S'_{12} \\
&= (4m^2n^2 - 4m^2n^2)S_{11} - 4(m^4 + n^4 + 2m^2n^2)S_{12} \\
&\quad + 4(mn^3 - m^3n - mn^3 + m^3n)S_{16} + (4m^2n^2 - 4m^2n^2)S_{22} \\
&\quad + 4(m^3n - mn^3 - m^3n + mn^3)S_{26} + \left[(m^2 - n^2)^2 + 4m^2n^2\right]S_{66} \\
&= S_{66} - 4S_{12}
\end{aligned} \tag{147}$$

$$\begin{aligned}
I_3 &= S'_{44} + S'_{55} \\
&= (m^2 + n^2)S_{44} + (2mn - 2mn)S_{45} + (m^2 + n^2)S_{55} \\
&= S_{44} + S_{55}
\end{aligned} \tag{148}$$

$$\begin{aligned}
I_4 &= S'_{23} + S'_{13} \\
&= (m^2 + n^2)S_{13} + (m^2 + n^2)S_{23} + (mn - mn)S_{36} \\
&= S_{13} + S_{23}
\end{aligned} \tag{149}$$

$$\begin{aligned}
I_5 &= S'^2_{34} + S'^2_{35} \\
&= (m^2 + n^2)S^2_{34} + (m^2 + n^2)S^2_{35} \\
&= S^2_{34} + S^2_{35}
\end{aligned} \tag{150}$$

$$I_6 = S'_{33} = S_{33} \tag{151}$$

An anisotropic body is subjected to hydrostatic pressure so that

$$\sigma_1 = \sigma_2 = \sigma_3 = p$$

and

$$\sigma_4 = \sigma_5 = \sigma_6 = 0$$

The change in volume is the sum of the normal strains; i.e.,

$$\delta V / V = e_1 + e_2 + e_3$$

The shear strains $e_4$, $e_5$, and $e_6$ under hydrostatic pressure will not be zero for an anisotropic body, but they do not contribute to any change in volume. Adding the first three equations of the generalized Hooke's law,

$$e_1 = S_{11}\sigma_1 + S_{12}\sigma_2 + S_{13}\sigma_3$$

$$e_2 = S_{12}\sigma_1 + S_{22}\sigma_2 + S_{23}\sigma_3$$

$$e_3 = S_{13}\sigma_1 + S_{23}\sigma_2 + S_{33}\sigma_3$$

$$\delta V/V = \left[S_{11} + S_{22} + S_{33} + 2(S_{12} + S_{23} + S_{31})\right] p$$

Thus, the compressibility of an anisotropic material is

$$pV/\delta V = 1/\left[S_{11} + S_{22} + S_{33} + 2(S_{12} + S_{23} + S_{31})\right] \tag{152}$$

$$= 1/(I_1 + I_6 + 2I_4) \tag{153}$$

The compressibility is also invariant. Similar invariants for the stiffness matrix can be established immediately, as follows:

$$\begin{aligned}
J_1 &= C'_{11} + C'_{22} + 2C'_{12} = C_{11} + C_{22} + 2C_{12} \\
J_2 &= C'_{66} - C'_{12} = C_{66} - C_{12} \\
J_3 &= C'_{44} + C'_{55} = C_{44} + C_{55} \\
J_4 &= C'_{23} + C'_{13} = C_{23} + C_{13} \\
J_5 &= {C'_{34}}^2 + {C'_{35}}^2 = C_{34}^2 + C_{35}^2 \\
J_6 &= C'_{33} = C_{33}
\end{aligned} \tag{154}$$

# SECTION III

## ELASTIC SYMMETRIES AND ENGINEERING CONSTANTS

The compliance and stiffness matrices in 3-space contain 21 independent components. If a material contains symmetry properties with respect to certain directions, which can be described in terms of coordinate transformations, the number of independent components will reduce. Ultimately, if a material is isotropic, there are only two independent constants. In this section, a few commonly encountered material symmetries and the relations between $S_{ij}$ or $C_{ij}$ and the engineering constants for various materials will be examined.

### ELASTIC SYMMETRIES

A triclinic material is the most general anisotropic material where all 21 elastic constants are independent. A fourth-rank tensor in 3-space will have 81 components. If both stress and strain are symmetric tensors, the 81 components can be represented by 36 independent components. If the stiffness and compliance matrices are symmetric, which can be demonstrated by assuming the existence of appropriate elastic potentials, then only 21 of the 36 components are independent.

Further reduction in the number of independent components can be introduced if additional material symmetry exists. One of the simplest forms of symmetry is the monoclinic material which possesses one plane of symmetry. Let plane x-y, or equivalently, z = 0, be a plane of symmetry; then the properties at +z are equal to those at -z. If a coordinate transformation of

$$t_{ij} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \tag{155}$$

will leave the $S_{ij}$ and $C_{ij}$ intact, the material by definition is a monoclinic material. The stress and strain components will transform with results very similar to those shown in Equations 94 through 101. Both the stress and strain components after the transformation will be as follows:

$$C'_{11} = C_{11}, \quad C'_{22} = C_{22}, \quad C'_{33} = C_{33},$$
$$C'_{23} = -C_{23}, \quad C'_{31} = -C_{31}, \quad C'_{12} = C_{12}$$

where $C_{ij}$, following Equations 94 through 101, is a typical second-rank tensor, and is not the stiffness matrix in the contracted notation. In the contracted notation, only the following components of stress and strain change signs:

$$\begin{aligned} \sigma'_4 &= -\sigma_4, \quad \sigma'_5 = -\sigma_5 \\ e'_4 &= -e_4, \quad e'_5 = -e_5 \end{aligned} \tag{156}$$

Compare the first equation of the generalized Hooke's law written in the new and old coordinates:

$$\sigma_1' = C_{11}e_1' + C_{12}e_2' + C_{13}e_3' + C_{14}e_4' + C_{15}e_5' + C_{16}e_6'$$

$$\sigma_1 = C_{11}e_1 + C_{12}e_2 + C_{13}e_3 - C_{14}e_4 - C_{15}e_5 + C_{16}e_6$$

If $\sigma_1 = \sigma_1'$, it is necessary that

$$C_{14} = C_{15} = 0 \tag{157}$$

By considering the remaining five equations in the generalized Hooke's law, one can show that

$$C_{24} = C_{25} = C_{34} = C_{35} = C_{64} = C_{65} = 0. \tag{158}$$

The compliance matrix must allow the same components to vanish as those shown in Equations 157 and 158. The number of independent components for $S_{ij}$ and $C_{ij}$ reduce from 21 to 13 for monoclinic materials, as follows:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & 0 & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ & & & S_{44} & S_{45} & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix} \tag{159}$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{22} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \tag{160}$$

Since both $S_{ij}$ and $C_{ij}$ are symmetric, the lower half of the matrices is not shown. The components shown above correspond to a monoclinic material with $z = 0$ as the plane of symmetry.

If a different plane of symmetry exists, say x = 0, the nonzero components of $S_{ij}$ and $C_{ij}$ will be different from those shown above. The $S_{ij}$ matrix will be:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & 0 & 0 \\ & S_{22} & S_{23} & S_{24} & 0 & 0 \\ & & S_{33} & S_{34} & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{55} & S_{56} \\ & & & & & S_{66} \end{bmatrix} \qquad (161)$$

The number of independent components for all monoclinic materials remain at 13 irrespective of the orientation of the symmetry plane.

If an anisotropic material possesses two orthogonal planes of symmetry, say, x = 0 and z = 0, the independent components must satisfy the $S_{ij}$ in Equations 159 and 161 simultaneously. This material is called orthotropic and must have the following compliance matrix:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix} \qquad (162)$$

The number of independent components reduces from 13 to 9. If a material has two orthogonal planes of symmetry, it will automatically have symmetry with respect to the third orthogonal plane.

If a material has a plane in which the property is isotropic, this is called a transversely isotropic material. Let us assume that the x-y (or 1-2) plane is isotropic, there is no preferred orientation in this plane. All properties will remain invariant under a rotation about the z-axis. The indices 1 and 2 in the $S_{ij}$ and $C_{ij}$ are interchangeable, thus:

$$S_{11} = S_{22}, \quad C_{11} = C_{22}$$

$$S_{13} = S_{23}, \quad C_{13} = C_{23}$$

And, the shear moduli between the z-direction and the isotropic plane (x-y) must be equal, thus

$$S_{44} = S_{55}, \quad C_{44} = C_{55}$$

Both $S_{11}$ and $S_{12}$ must be invariant. From the transformation equation of $S_{12}$, listed in Equation 140

$$\begin{aligned} S'_{12} &= m^2n^2S_{11} + (m^4+n^4)S_{12} + m^2n^2S_{22} - m^2n^2S_{66} \\ &= S_{12} + m^2n^2(S_{11} + S_{22} - 2S_{12} - S_{66}) \end{aligned}$$

Since

$$S'_{12} = S_{12} \quad \text{and} \quad S_{11} = S_{22},$$

then

$$S_{66} = 2(S_{11} - S_{12}) \tag{163}$$

In a similar way, one can show

$$C_{66} = (C_{11} - C_{12})/2 \tag{164}$$

A transversely isotropic material relative to the 3-axis will have the following compliance and stiffness matrices:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & 0 \\ & S_{11} & S_{13} & 0 & 0 & 0 \\ & & S_{33} & 0 & 0 & 0 \\ & & & S_{44} & 0 & 0 \\ & & & & S_{44} & 0 \\ & & & & & 2(S_{11} - S_{12}) \end{bmatrix} \tag{165}$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{11} & C_{13} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & & & & C_{44} & 0 \\ & & & & & (C_{11} - C_{12})/2 \end{bmatrix} \tag{166}$$

If the isotropic plane is in the 2-3 plane or with respect to the 1-axis, the compliance matrix becomes

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{22} & S_{23} & 0 & 0 & 0 \\ & & S_{22} & 0 & 0 & 0 \\ & & & 2(S_{22} - S_{23}) & 0 & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{55} \end{bmatrix} \tag{167}$$

The number of independent elastic constants for this material is five.

In the case of isotropic materials, indices 1, 2, and 3, and 4, 5, and 6 are interchangeable; thus,

$$S_{11} = S_{22} = S_{33}, \quad C_{11} = C_{22} = C_{33}$$
$$S_{23} = S_{31} = S_{12}, \quad C_{23} = C_{31} = C_{12}$$
$$S_{44} = S_{55} = S_{66}, \quad C_{44} = C_{55} = C_{66}$$

In addition,

$$S_{44} = 2(S_{11} - S_{12}), \quad C_{44} = (C_{11} - C_{12})/2$$

The final $S_{ij}$ and $C_{ij}$ for isotropic materials are

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ & S_{11} & S_{12} & 0 & 0 & 0 \\ & & S_{11} & 0 & 0 & 0 \\ & & & 2(S_{11}-S_{12}) & 0 & 0 \\ & & & & 2(S_{11}-S_{12}) & 0 \\ & & & & & 2(S_{11}-S_{12}) \end{bmatrix} \qquad (168)$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & (C_{11}-C_{12})/2 & 0 & 0 \\ & & & & (C_{11}-C_{12})/2 & 0 \\ & & & & & (C_{11}-C_{12})/2 \end{bmatrix} \qquad (169)$$

There are only two independent elastic constants.

## ENGINEERING CONSTANTS

Engineering constants usually refer to Young's moduli, Poisson's ratios, and shear moduli. These material constants can be measured from simple tests, such as the uniaxial tension or pure torsion tests. The constants are therefore more familiar and understandable than the components of $S_{ij}$ and $C_{ij}$. The relationship between these components and the engineering constants will be established in this section.

Since most simple tests are performed with a known imposed load or stress, the resulting displacement or strain is measured. The former is the independent variable; the latter, the dependent variable. Thus, the components of the compliance matrix $S_{ij}$ can be more explicitly determined than those of $C_{ij}$. The following relations between the components of $S_{ij}$ and the

engineering constants can be established immediately from the nature of uniaxial and pure shear tests:

$$
\begin{aligned}
S_{11} &= 1/E_{11}, & S_{22} &= 1/E_{22}, & S_{33} &= 1/E_{33} \\
S_{12} &= -\nu_{12}/E_{11}, & S_{23} &= -\nu_{23}/E_{22}, & S_{31} &= -\nu_{31}/E_{33} \\
&= -\nu_{21}/E_{22} & &= -\nu_{32}/E_{33} & &= -\nu_{13}/E_{11} \\
S_{66} &= 1/G_{12}, & S_{55} &= 1/G_{13}, & S_{44} &= 1/G_{23} \\
S_{16} &= \eta_{16}/E_{11}, & S_{26} &= \eta_{26}/E_{22}, & S_{36} &= \eta_{36}/E_{33}
\end{aligned}
\tag{170}
$$

where the $\eta_{ij}$ are the shear coupling ratios. Other components of $S_{ij}$ can also be expressed in terms of engineering constants. But new and unfamiliar engineering constants must be invented and it is doubtful that a useful purpose is served by forcing a complete equivalence between the engineering constants and the components of $S_{ij}$. In fact, only orthotropic, transversely isotropic, and isotropic materials are being investigated in these notes. The triclinic and monoclinic materials are being omitted to avoid using unfamiliar engineering constants. The shear coupling ratios often appear in two-dimensional problems.

For an orthotropic material, one can express the components of $C_{ij}$ in terms of the engineering constants. This can be done in a straightforward manner by making a substitution of Equation 170 into an inverted $S_{ij}$. The resulting relations are listed below:

$$
\begin{aligned}
C_{11} &= (1-\nu_{23}\nu_{32})VE_{11} \\
C_{22} &= (1-\nu_{31}\nu_{13})VE_{22} \\
C_{33} &= (1-\nu_{12}\nu_{21})VE_{33} \\
C_{12} &= (\nu_{21}+\nu_{23}\nu_{31})VE_{11} = (\nu_{12}+\nu_{13}\nu_{32})VE_{22} \\
C_{13} &= (\nu_{31}+\nu_{21}\nu_{32})VE_{11} = (\nu_{13}+\nu_{23}\nu_{12})VE_{33} \\
C_{23} &= (\nu_{32}+\nu_{12}\nu_{31})VE_{22} = (\nu_{23}+\nu_{21}\nu_{13})VE_{33} \\
C_{44} &= G_{23} \\
C_{55} &= G_{31} \\
C_{66} &= G_{12}
\end{aligned}
\tag{171}
$$

where

$$
V = (1-\nu_{12}\nu_{21}-\nu_{23}\nu_{32}-\nu_{31}\nu_{13}-2\nu_{12}\nu_{23}\nu_{31})^{-1} \tag{172}
$$

The relations between $C_{ij}$ and the engineering constants are considerably more complicated than those for $S_{ij}$. Poisson's ratios are responsible for the complicated relations above. If

all Poisson's ratios are zero, which means that there is no coupling between the normal strains or stresses, then

$$C_{11} = E_{11}, \quad C_{22} = E_{22}, \quad C_{33} = E_{33},$$
$$C_{12} = C_{13} = C_{32} = 0.$$

If the material is transversely isotropic with plane 2-3 as the isotropic plane, the components of Equation 167 can be related to the engineering constants as follows:

$$\begin{aligned}
S_{11} &= 1/E_{11}, \qquad S_{22} = S_{33} = 1/E_{22} \\
S_{13} &= S_{12} = -\nu_{12}/E_{11} = -\nu_{21}/E_{22} \\
S_{23} &= -\nu_{23}/E_{22} \\
S_{44} &= 2(1+\nu_{23})/E_{22} \\
S_{55} &= S_{66} = 1/G_{12}
\end{aligned} \tag{173}$$

The components of $C_{ij}$ are

$$\begin{aligned}
C_{11} &= (1-\nu_{23}^2)VE_{11} \\
C_{22} &= C_{33} = (1-\nu_{12}\nu_{21})VE_{22} \\
C_{12} &= C_{13} = \nu_{21}(1+\nu_{23})VE_{11} = \nu_{12}(1+\nu_{23})VE_{22} \\
C_{23} &= (\nu_{23}+\nu_{12}\nu_{21})VE_{22} \\
C_{44} &= (C_{22}-C_{23})/2 = (1-\nu_{23}-2\nu_{12}\nu_{21})VE_{22}/2 \\
C_{55} &= C_{66} = G_{12}
\end{aligned} \tag{174}$$

where

$$\begin{aligned}
V &= (1-2\nu_{12}\nu_{21}-\nu_{23}^2-2\nu_{12}\nu_{21}\nu_{23})^{-1} \\
&= \left[(1+\nu_{23})(1-\nu_{23}-2\nu_{12}\nu_{21})\right]^{-1}
\end{aligned} \tag{175}$$

Simplification of Equation 174 can be made by using Equation 175. If the material is isotropic,

$$\begin{aligned}
S_{11} &= S_{22} = S_{33} = 1/E \\
S_{12} &= S_{13} = S_{23} = -\nu/E \\
S_{44} &= S_{55} = S_{66} = 1/G = 2(1+\nu)/E
\end{aligned} \tag{176}$$

The components of $C_{ij}$ are:

$$
\begin{aligned}
C_{11} &= C_{22} = C_{33} = \frac{(1-\nu)E}{(1+\nu)(1-2\nu)} \\
C_{12} &= C_{13} = C_{23} = \frac{\nu E}{(1+\nu)(1-2\nu)} \\
C_{44} &= C_{55} = C_{66} = G = \frac{E}{2(1+\nu)}
\end{aligned}
\tag{177}
$$

In this section, the components of $S_{ij}$ and $C_{ij}$ for orthotropic, transversely isotropic, and isotropic materials are expressed in terms of commonly encountered engineering constants. The components of $S_{ij}$ have simpler relations than those of $C_{ij}$ with the engineering constants.

## TRANSFORMED $S_{ij}$

The nonzero components of $S_{ij}$ for monoclinic, orthotropic, and transversely isotropic materials in coordinate systems other than their material symmetry axes can easily be established from the transformation equation listed in Equations 140 through 145. The transformation being investigated is restricted to a proper rotation about the 3 axis. All components which have primes designate transformed components.

In the monoclinic case with the symmetry plane containing the 1-axis and the 2-axis, Equation 159 represents the independent components of $S_{ij}$. A rotation about the 3-axis would result in the following nonzero components:

$$
S'_{ij} = \begin{bmatrix}
S'_{11} & S'_{12} & S'_{13} & 0 & 0 & S'_{16} \\
 & S'_{22} & S'_{23} & 0 & 0 & S'_{26} \\
 & & S'_{33} & 0 & 0 & S'_{36} \\
 & & & S'_{44} & S'_{45} & 0 \\
 & & & & S'_{55} & 0 \\
 & & & & & S'_{66}
\end{bmatrix}
\tag{178}
$$

The number of nonzero components (20) of Equation 178 does not differ from that of Equation 159 because the axis of rotation coincides with the normal to the symmetry plane. The $S_{ij}$ in Equation 161, however, has a plane of symmetry containing the 2-axis and 3-axis. When

the same transformation as above is applied to this monoclinic material, the resulting $S_{ij}$ becomes

$$S'_{ij} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{13} & S'_{14} & S'_{15} & S'_{16} \\ & S'_{22} & S'_{23} & S'_{24} & S'_{25} & S'_{26} \\ & & S'_{33} & S'_{34} & S'_{35} & S'_{36} \\ & & & S'_{44} & S'_{45} & S'_{46} \\ & & & & S'_{55} & S'_{56} \\ & & & & & S'_{66} \end{bmatrix} \tag{179}$$

There are 36 nonzero components, as compared with 20 in the principal direction of $S_{ij}$. But this is not a triclinic material because of the 36 transformed components, only 13 are independent. To distinguish between the $S_{ij}$ in Equations 159 or 178 and 179, the former may be called special monoclinic, the latter, general monoclinic. The special monoclinic refers to the $S_{ij}$ in its principal axes or the material symmetry axes. All monoclinic materials have only 13 independent components; and only the special monoclinic material has 20 nonzero components, as shown in Equation 159 or 178. A special orthotropic material is shown in Equation 162. Using the transformation Equations 140 through 145, the general orthotropic material can be shown to have the following $S_{ij}$:

$$S'_{ij} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{13} & 0 & 0 & S'_{16} \\ & S'_{22} & S'_{23} & 0 & 0 & S'_{26} \\ & & S'_{33} & 0 & 0 & S'_{36} \\ & & & S'_{44} & S'_{45} & 0 \\ & & & & S'_{55} & 0 \\ & & & & & S'_{66} \end{bmatrix} \tag{180}$$

There are 20 nonzero components, of which 9 are independent. A general orthotropic material has the appearance of a special monoclinic material.

A special transversely isotropic material relative to the 3-axis is shown in Equation 165. For this material,

$$S_{11} = S_{22}, \quad S_{66} = 2(S_{11} - S_{12})$$

$$S_{13} = S_{23}, \quad S_{44} = S_{55}$$

By substituting these conditions into the transformation equations in Equation 140, 141, and 142 one can show that

$$S'_{16} = S'_{26} = S'_{36} = S'_{45} = 0$$

The general transversely isotropic material corresponding to Equation 165 is

$$S_{ij} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{13} & 0 & 0 & 0 \\ & S'_{11} & S'_{13} & 0 & 0 & 0 \\ & & S'_{33} & 0 & 0 & 0 \\ & & & S'_{44} & 0 & 0 \\ & & & & S'_{44} & 0 \\ & & & & & 2(S'_{11} - S'_{12}) \end{bmatrix} \qquad (181)$$

Since the rotation is in the isotropic plane, Equation 181 is the same as 165. An analogous situation occurred between Equations 178 and 159 for the monoclinic material.

For a special transversely isotropic material with a different isotropic plane, e.g., the 2-3 plane as shown in Equation 167, the corresponding general transversely isotropic material will be

$$S'_{ij} = \begin{bmatrix} S'_{11} & S'_{12} & S'_{13} & 0 & 0 & S'_{16} \\ & S'_{22} & S'_{23} & 0 & 0 & S'_{26} \\ & & S'_{33} & 0 & 0 & S'_{36} \\ & & & S'_{44} & S'_{45} & 0 \\ & & & & S'_{55} & 0 \\ & & & & & S'_{66} \end{bmatrix} \qquad (182)$$

There are 20 nonzero components that are similar to the special monoclinic material, but the number of independent components for a transversely isotropic material remains at 5.

For isotropic material, one can substitute the relation in Equation (176) and readily show that the $S_{ij}$ remains the same as Equation 168, which has 12 nonzero components (2 independent components).

# SECTION IV

# TWO-DIMENSIONAL COMPOSITES

Two-dimensional formulations are of particular interest in the study of composite materials. In this section, the assumptions of plane strain and plane stress, which represent two special two-dimensional problems, will be described. Laminated composites as special cases of plane stress will also be discussed. All two-dimensional problems are reduced from a three-dimensional special monoclinic material with z = 0 as its symmetry plane. Any rotation about the z-axis will not make the special monoclinic into a general monoclinic material; this is shown in Equation 178. The use of this material which possesses 20 nonzero (13 independent components will include, as special cases, the general orthotropic and general transversely isotropic materials, shown in Equations 180 and 182, respectively.

## PLANE STRAIN

A state of plane strain is obtained by assuming that

$$\begin{aligned} u_1 &= u_1(x_1, x_2) \\ u_2 &= u_2(x_1, x_2) \\ u_3 &= u_0 \end{aligned} \tag{183}$$

where $u_i$ are the displacements along 1, 2, and 3 axes and $u_0$ is a constant. From the strain displacement equation, one obtains

$$e_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

When this is expanded,

$$\begin{aligned} e_3 &= e_{33} = \frac{\partial u_3}{\partial x_3} = 0 \\ e_4 &= 2e_{23} = \frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2} = 0 \\ e_5 &= 2e_{31} = \frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1} = 0 \end{aligned} \tag{184}$$

For a monoclinic material with the following compliance and stiffness matrices:

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & 0 & 0 & S_{16} \\ & S_{22} & S_{23} & 0 & 0 & S_{26} \\ & & S_{33} & 0 & 0 & S_{36} \\ & & & S_{44} & S_{45} & 0 \\ & & & & S_{55} & 0 \\ & & & & & S_{66} \end{bmatrix} \tag{185}$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & C_{16} \\ & C_{22} & C_{23} & 0 & 0 & C_{26} \\ & & C_{33} & 0 & 0 & C_{36} \\ & & & C_{44} & C_{45} & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix} \tag{186}$$

it can be shown by substituting Equation 184 into the generalized Hooke's law in terms of the stiffness matrix that

$$\begin{aligned} \sigma_4 &= \sigma_5 = 0 \\ \sigma_3 &= C_{31}e_1 + C_{32}e_2 + C_{36}e_6 \end{aligned} \tag{187}$$

and from the generalized Hooke's law in terms of the compliance matrix,

$$\sigma_3 = -\frac{1}{S_{33}}(S_{31}\sigma_1 + S_{32}\sigma_2 + S_{36}\sigma_6)$$

Since $\sigma_3$ is now dependent on the other stress components, it can be eliminated from the generalized Hooke's law. The results are:

$$e_i = R_{ij}\sigma_j \,, \tag{188}$$

where

$$R_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}, \quad i, j = 1, 2, 6$$

$R_{ij}$ represents reduced constants derived from $S_{ij}$. It is applicable for the case of plane strain imposed on a special monoclinic material. The components of $C_{ij}$ remain unchanged, so that the generalized Hooke's law for plane strain is

$$\sigma_i = C_{ij} e_j \tag{189}$$

where i, j = 1, 2, and 6. Where i, j = 3, 4, and 5, the Hooke's law is expressed by Equations 187.

For a special orthotropic material, under a state of plane strain, the compliance and stiffness matrices can be written down from the results shown in equation 162, so that

$$S_{16} = S_{26} = S_{36} = C_{16} = C_{26} = 0$$

Thus,

$$R_{16} = R_{26} = 0.$$

$$R_{ij} = \begin{bmatrix} S_{11} - \dfrac{S_{13}^2}{S_{33}} & S_{12} - \dfrac{S_{13}S_{23}}{S_{33}} & 0 \\ & S_{22} - \dfrac{S_{23}^2}{S_{33}} & 0 \\ & & S_{66} \end{bmatrix} \tag{190}$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{22} & 0 \\ & & C_{66} \end{bmatrix} \tag{191}$$

For a special transversely isotropic material shown in Equation 167,

$$R_{ij} = \begin{bmatrix} S_{11} - \frac{S_{12}^2}{S_{22}} & S_{12} - \frac{S_{12}S_{23}}{S_{22}} & 0 \\ & S_{22} - \frac{S_{23}^2}{S_{22}} & 0 \\ & & S_{66} \end{bmatrix} \tag{192}$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{22} & 0 \\ & & C_{66} \end{bmatrix} \tag{193}$$

Finally, for isotropic material,

$$R_{ij} = \begin{bmatrix} S_{11} - \frac{S_{12}^2}{S_{11}} & S_{12} - \frac{S_{12}^2}{S_{11}} & 0 \\ & S_{11} - \frac{S_{12}^2}{S_{11}} & 0 \\ & & 2(S_{11}-S_{12}) \end{bmatrix} \tag{194}$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & 0 \\ & C_{11} & 0 \\ & & \frac{1}{2}(C_{11}-C_{12}) \end{bmatrix} \tag{195}$$

The reduced compliance matrix in terms of the engineering constants can be obtained by direct substitution of Equation 170 into 190, 192, and 194, and the results are

1) Orthotropic material:

$$\begin{aligned} R_{11} &= (1 - \nu_{13}\nu_{31})/E_{11} \\ R_{22} &= (1 - \nu_{23}\nu_{32})/E_{22} \\ R_{12} &= -(\nu_{12} + \nu_{13}\nu_{32})/E_{11} \\ &= -(\nu_{21} + \nu_{31}\nu_{23})/E_{22} \\ R_{66} &= S_{66} = 1/G_{12} \end{aligned} \tag{196}$$

2) Transversely isotropic material:

$$\begin{aligned} R_{11} &= (1 - \nu_{12}\nu_{21})/E_{11} \\ R_{22} &= (1 - \nu_{23}^2)/E_{22} \\ R_{12} &= -\nu_{12}(1 + \nu_{23})/E_{11} \\ &= -\nu_{21}(1 + \nu_{23})/E_{22} \\ R_{66} &= S_{66} = 1/G_{12} \end{aligned} \tag{197}$$

3) Isotropic material:

$$\begin{aligned} R_{11} &= R_{22} = (1 - \nu^2)/E \\ R_{12} &= -\nu(1 + \nu)/E \\ R_{66} &= 2(1 + \nu)/E \end{aligned} \tag{198}$$

The stiffness matrix in terms of the engineering constants are exactly the same as those for the three-dimensional case. For the special orthotropic material, they are shown in Equations 171; the special transversely isotropic material Equation 174; and the isotropic material, Equation 177.

It can be shown that

$$R_{ij}C_{jk} = \delta_{ik} \tag{199}$$

which indicates, as expected, that the reduced compliance matrix is the inverse of the stiffness matrix.

PLANE STRESS

Another two-dimensional problem can be formulated by assuming

$$\sigma_3 = \sigma_4 = \sigma_5 = 0 \tag{200}$$

From the $S_{ij}$ of the special monoclinic material shown in Equation 185,

$$\begin{aligned} e_4 &= e_5 = 0 \\ e_3 &= S_{31}\sigma_1 + S_{32}\sigma_2 + S_{36}\sigma_6 \end{aligned} \tag{201}$$

The generalized Hooke's law becomes

$$e_i = S_{ij}\sigma_j \text{ , where } i,j = 1,\ 2,\ 6 \tag{202}$$

From $C_{ij}$ shown in Equation 186

$$e_3 = -\frac{1}{C_{33}}(C_{31}e_1 + C_{32}e_2 + C_{36}e_6) \tag{203}$$

The $e_3$ is not an independent component. This is analogous to $\sigma_3$ being a dependent component in the case of plane strain. Substitute Equation 203 into the generalized Hooke's law in terms of $C_{ij}$; then

$$\sigma_i = Q_{ij}e_j \tag{204}$$

where

$$\begin{aligned} Q_{ij} &= \text{reduced stiffness matrix} \\ &= C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \end{aligned} \tag{205}$$

and

$$i,j = 1,\ 2,\ 6$$

Thus, for plane stress, the $S_{ij}$ remains the same as a three-dimensional material, whereas the $C_{ij}$ must be replaced by $Q_{ij}$.

For a special orthotropic material,

$$\begin{aligned} S_{16} = S_{26} = S_{36} = S_{45} = 0 \\ C_{16} = C_{26} = C_{36} = S_{45} = 0 \end{aligned} \tag{206}$$

The $S_{ij}$ and $Q_{ij}$ for a state of plane stress are

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{22} & 0 \\ & & S_{66} \end{bmatrix} \tag{207}$$

$$Q_{ij} = \begin{bmatrix} C_{11} - \dfrac{C_{13}^2}{C_{33}} & C_{12} - \dfrac{C_{13}C_{23}}{C_{33}} & 0 \\ & C_{22} - \dfrac{C_{23}^2}{C_{33}} & 0 \\ & & C_{66} \end{bmatrix} \tag{208}$$

For a transversely isotropic material,

$$\begin{aligned} S_{22} &= S_{33}, \quad S_{12} = S_{13} \\ C_{22} &= C_{33}, \quad C_{12} = C_{13} \end{aligned} \tag{209}$$

Then, $S_{ij}$ is the same as Equation 207, but $Q_{ij}$ is

$$Q_{ij} = \begin{bmatrix} C_{11} - \dfrac{C_{12}^2}{C_{22}} & C_{12} - \dfrac{C_{12}C_{23}}{C_{22}} & 0 \\ & C_{22} - \dfrac{C_{23}^2}{C_{22}} & 0 \\ & & C_{66} \end{bmatrix} \tag{210}$$

Finally, for an isotropic material,

$$\begin{aligned} S_{11} &= S_{22}, \quad S_{12} = S_{23}, \quad S_{66} = 2(S_{11} - S_{12}) \\ C_{11} &= C_{22}, \quad C_{12} = C_{23}, \quad C_{66} = (C_{11} - C_{12})/2 \end{aligned} \tag{211}$$

Then, $S_{ij}$ and $Q_{ij}$ are

$$S_{ij} = \begin{bmatrix} S_{11} & S_{12} & 0 \\ & S_{11} & 0 \\ & & 2(S_{11}-S_{12}) \end{bmatrix} \tag{212}$$

$$Q_{ij} = \begin{bmatrix} C_{11} - \frac{C_{12}^2}{C_{11}} & C_{12} - \frac{C_{12}^2}{C_{11}} & 0 \\ & C_{11} - \frac{C_{12}^2}{C_{11}} & 0 \\ & & (C_{11}-C_{12})/2 \end{bmatrix} \tag{213}$$

In terms of the engineering constants, the $S_{ij}$ for all three materials follows Equation 170. The $Q_{ij}$ can be obtained by substituting Equation 172 into Equation 208 for the orthotropic material; Equation 174 into 210 for the transversely isotropic material; and finally, Equation 177 into 213 for the isotropic materia.. The resulting relations are listed below:

1) Orthotropic materials:

$$\begin{aligned} Q_{11} &= E_{11}/(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_{22}/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{21}E_{11}/(1-\nu_{12}\nu_{21}) = \nu_{12}E_{22}/(1-\nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \end{aligned} \tag{214}$$

2) Transversely isotropic materials:

$$\begin{aligned} Q_{11} &= E_{11}/(1-\nu_{12}\nu_{21}) \\ Q_{22} &= E_{22}/(1-\nu_{12}\nu_{21}) \\ Q_{12} &= \nu_{21}E_{11}/(1-\nu_{12}\nu_{21}) = \nu_{12}E_{22}/(1-\nu_{12}\nu_{21}) \\ Q_{66} &= G_{12} \end{aligned} \tag{215}$$

The $Q_{ij}$ for the transversely isotropic material is the same as that of the orthotropic material. Engineering constants associated with the 3-direction, e.g., $\nu_{13}$ and $\nu_{23}$, do not appear in the

plane stress cases. These constants, however, appear in the plane strain case. Thus, under a state of plane stress, orthotropic and transversely isotropic materials are identical, but under plane strain, and three-dimensional problems in general, these two materials are of course different.

3) Isotropic materials:

$$Q_{11} = Q_{22} = E/(1-\nu^2)$$
$$Q_{12} = \nu E/(1-\nu^2) \qquad (216)$$
$$Q_{66} = E/2(1+\nu)$$

Again, it can be shown that

$$Q_{ij}S_{jk} = \delta_{ik} \qquad (217)$$

which indicates that $Q_{ij}$ is the inverse of $S_{ij}$.

COMPARISON OF PLANE STRAIN AND PLANE STRESS

In two-dimensional problems, modifications to the compliance and stiffness matrices may be necessary. The appropriate Hooke's laws are

1) Plane Strain:

$$\sigma_i = C_{ij}e_j$$
$$e_i = R_{ij}\sigma_j \qquad (218)$$

where

$$R_{ij} = S_{ij} - \frac{S_{i3}S_{j3}}{S_{33}}$$
$$i,j = 1, 2, 6$$

2) Plane Stress:

$$\sigma_i = Q_{ij}e_j$$
$$e_i = S_{ij}\sigma_j \qquad (219)$$

where

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}}$$
$$i,j = 1, 2, 6$$

In microscopic mechanics analysis of composite materials, it is a common practice to solve an inclusion problem in plane strain and plane stress, from which the effective transverse stiffness $E_{22}$, e.g., can be predicted. Let us assume that a transverse load $\sigma_2$ is imposed, while $\sigma_1 = \sigma_6 = 0$. In a plane strain case, from Equation 218

$$e_2 = R_{22}\sigma_2 \tag{220}$$

Thus, the effective stiffness in the transverse direction (the 2-direction) is

$$\sigma_2 / e_2 = 1/R_{22}$$

In terms of engineering constants, from Equations 196 and 197,

$$\sigma_2 / e_2 = E_{22}/(1 - \nu_{23}\nu_{32}) \tag{221}$$

or

$$= E_{22}/(1 - \nu_{23}^2) \tag{222}$$

for orthotropic or transversely isotropic materials, respectively. For a plane stress case, with $\sigma_2 \neq 0$, $\sigma_1 = \sigma_6 = 0$,

$$\sigma_2 / e_2 = 1/S_{22} = E_{22} \tag{223}$$

This is true for both orthotropic and transversely isotropic materials. A comparison of Equations 221 or 222 with Equation 223, shows that the Poisson ratio associated with the 3-direction enters the plane strain analysis but not the plane stress case.

## LAMINATED COMPOSITES

Laminated composites to be considered in this subsection consist of layers of thin orthotropic plates bonded together. Each layer may have arbitrary thickness and orientation of its material symmetry axes. In general, each constituent layer is a general orthotropic material. Assuming that the coefficients of thermal expansion or contraction are also orthotropic, the three-dimensional generalized Hooke's law for each layer may be modified as follows:

$$e_i = S_{ij}\sigma_j + a_i T, \quad i,j = 1, 2, \ldots, 6 \tag{224}$$

where $a_i$ = thermal expansion matrix, T = temperature. The first term represents mechanical strain, and the second, thermal strain. The thermal expansion matrix is a second-rank tensor, as indicated by the single index in the contracted notation. For an orthotropic material, the independent components of a second-rank tensor are, for example, $a_1$, $a_2$, and $a_3$, while $a_4 = a_5 = a_6 = 0$. In a general orthotropic material with a rotation about the 3-axis, $a_6$ will not be zero. Equation 224 can be inverted to have the following form:

$$\sigma_i = C_{ij}(e_j - a_j T), \quad i,j = 1, 2, \ldots, 6 \tag{225}$$

If we assume that each constituent layer is under a state of plane stress, which is reasonable for thin plates subjected to in-plane (plane 1-2) stresses,

$$\sigma_3 = \sigma_4 = \sigma_5 = 0 \tag{226}$$

From Equations 224 and 225, we can show that

$$e_4 = e_5 = 0$$

$$e_3 - \alpha_3 T = -\frac{C_{31}}{C_{33}}(e_1 - \alpha_1 T) - \frac{C_{32}}{C_{33}}(e_2 - \alpha_2 T) - \frac{C_{36}}{C_{33}}(e_6 - \alpha_6 T) \tag{227}$$

From this, reduced constants $Q_{ij}$ can be obtained, following a similar derivation described in the last subsection. The generalized Hooke's law in a state of plane stress including the thermal effect is:

$$\sigma_i = Q_{ij}(e_j - \alpha_j T), \quad i,j = 1, 2, 6 \tag{228}$$

where

$$Q_{ij} = C_{ij} - \frac{C_{i3}C_{j3}}{C_{33}} \tag{229}$$

and their relations to engineering constants are shown in Equation 214 or 215.

If a laminated composite is thin and the deflection of the composite plate is kept small relative to its thickness, it is reasonable to assume that normals to the middle surface are nondeformable. With this assumption,

$$e_i = e_i^o + zk_i, \quad i = 1, 2, 6 \tag{230}$$

where $e_i^o$ = in-plane strain and $k_i$ = curvature with the following definitions:

$$e_1^o = u_{1,1} \qquad k_1 = u_{3,11}$$

$$e_2^o = u_{2,2} \qquad k_2 = u_{3,22}$$

$$e_6^o = u_{1,2} + u_{2,1} \qquad k_6 = 2u_{3,12}$$

To be consistent with the use of engineering strains, both the in-plane strain and curvature are expressed in engineering quantities, as opposed to tensorial quantities. This is not a trivial point. The use of engineering strains and curvatures is preferred, particularly in the contracted notation. A number of symmetries, e.g., in $C_{ij}$, $S_{ij}$, $R_{ij}$, and $Q_{ij}$, and later in $A_{ij}$, $B_{ij}$, and $D_{ij}$ in Equation 236, can be maintained with the proper use of the contracted notation. As stated earlier in these notes, some authors have employed their own contracted notation and may have caused some unnecessary confusion.

The substitution of Equation 230 into 228 yields

$$\sigma_i = Q_{ij}(e_j^o + zk_j) - Q_{ij}\alpha_j T \tag{231}$$

In a thin homogeneous or laminated plate, it is convenient to deal with stress resultants and stress couples, which are defined as follows:

$$(N_i, M_i) = \int_{-h/2}^{h/2} \sigma_i (1, z)\,dz \tag{232}$$

The integration of Equation 231 gives

$$\overline{N}_i = N_i + N_i^T = A_{ij}e_j^o + B_{ij}k_j \tag{233}$$

$$\overline{M}_i = M_i + M_i^T = B_{ij}e_j^o + D_{ij}k_j \tag{234}$$

where

$$(N_i^T, M_i^T) = \int_{-h/2}^{h/2} Q_{ij}\alpha_j T(1, z)\,dz \tag{235}$$

$$(A_{ij}, B_{ij}, D_{ij}) = \int_{-h/2}^{h/2} Q_{ij}(1, z, z^2)\,dz \tag{236}$$

Equations 233 and 234 are the basic constitutive equations for laminated anisotropic plates subjected to small deflections. These equations are applicable to thin shells, if the radii of curvature of the shells are large in comparison with the shell thickness. The effect of temperature is taken into account by the equivalent thermal loadings, $N_i^T$ and $M_i^T$. The deformations induced by a temperature change are equal to those produced by applying the thermal loads.

The stress at any point in a laminated anisotropic body can be obtained by inverting Equations 233 and 234, which in matrix form, are:

$$\begin{bmatrix} \overline{N} \\ \overline{M} \end{bmatrix} = \begin{bmatrix} A & B \\ B & D \end{bmatrix} \begin{bmatrix} e^o \\ k \end{bmatrix} \tag{237}$$

$$\begin{bmatrix} e^o \\ \overline{M} \end{bmatrix} = \begin{bmatrix} A^* & B^* \\ H^* & D^* \end{bmatrix} \begin{bmatrix} \overline{N} \\ k \end{bmatrix} \tag{238}$$

$$\begin{bmatrix} e^o \\ k \end{bmatrix} = \begin{bmatrix} A' & B' \\ B' & D' \end{bmatrix} \begin{bmatrix} \overline{N} \\ \overline{M} \end{bmatrix} \tag{239}$$

where

$$\begin{aligned} A^* &= A^{-1} \\ B^* &= -A^{-1}B \\ H^* &= BA^{-1} \\ D^* &= D - BA^{-1}B \\ A' &= A^* - B^* D^{*-1} H^* \\ B' &= B^* D^{*-1} \\ D' &= D^{*-1} \end{aligned} \tag{240}$$

The substitution of Equation 239 into 230 yields

$$\begin{aligned} e_i &= e_i^o + z k_i \\ &= (A'_{ij} + zB'_{ij})\overline{N_j} + (B'_{ij} + zD'_{ij})\overline{M_j} \end{aligned} \tag{241}$$

From Equation 231, the stress components for the k-th layer of a laminated composite are

$$\begin{aligned} \sigma_i^{(k)} &= Q_{ij}^{(k)}(e_j - \alpha_j^{(k)}T) \\ &= Q_{ij}^{(k)}\left[(A'_{jk} + zB'_{jk})\overline{N}_k + (B'_{jk} + zD'_{jk})\overline{M}_k - \alpha_j^{(k)}T\right] \end{aligned} \tag{242}$$

$$\begin{aligned} &= Q_{ij}^{(k)}\left[(A'_{jk} + zB'_{jk})N_k + (B_{jk} + zD'_{jk})M_k\right] \\ &\quad + Q_{ij}^{(k)}\left[(A'_{jk} + zB'_{jk})\int Q_{kl}\alpha_l dz\right. \\ &\quad \left. + (B'_{jk} + zD'_{jk})\int Q_{kl}\alpha_l z dz - \alpha_j^{(k)}\right]T \end{aligned} \tag{243}$$

The last equation is derived by assuming a constant temperature in Equation 235. In this case the temperature effect is lumped into one term.

Matrices A, B, and D are the intrinsic properties of a laminated composite. They depend on the properties of each constituent layer $Q_{ij}^{(k)}$, the thickness $h^{(k)}$, and the stacking sequence of the layers.

If all layers are quasi-homogeneous, the integrations of Equation 236 can be replaced by summations, as follows:

$$A_{ij} = \sum_{k=1}^{n} Q_{ij}^{(k)}(h_{k+1} - h_k) \tag{244}$$

$$B_{ij} = \frac{1}{2}\sum_{k=1}^{n} Q_{ij}^{(k)}(h_{k+1}^2 - h_k^2) \tag{245}$$

$$D_{ij} = \frac{1}{3}\sum_{k=1}^{n} Q_{ij}^{(k)}(h_{k+1}^3 - h_k^3) \tag{246}$$

Thus, matrices A, B, and D are simple to determine for a laminated composite with a limited number of layers. But for the determination of stress and strain, from Equations 242 and 241, respectively, the prime matrices A', B', and D' are required. These matrices are obtained by the matrix inversion operations shown in Equation 240. The inversion of a 6 x 6 matrix is very difficult to do by hand. This unfortunate situation is unavoidable in the case of a general laminated composite where all 36 components are nonzero.

A considerable amount of simplification is possible if the B-matrix is identically zero. This occurs if the laminated composite is symmetrical with respect to the middle surface, the z = 0 plane. With B = 0, Equation 240 can be simplified, as follows:

$$\begin{aligned} A^* &= A^{-1} \\ B^* &= H^* = 0 \\ D^* &= D \\ A' &= A^{-1} \\ B' &= 0 \\ D' &= D^{-1} \end{aligned} \tag{247}$$

In this case, only the inversion of 3 x 3 matrices is involved and this is certainly manageable by hand computations.

A laminated anisotropic composite is governed by 18 independent constants. This number can be reduced if symmetry in the method of lamination and symmetry in the constituent layers exist. For a homogeneous isotropic plate, the number of independent constants reduces to 2. It is important to know that material properties should be referred to the components of A, B, and D matrices, or equivalently, the star or prime matrices. They cannot be expressed in terms of engineering constants, as homogeneous orthotropic or transversely isotropic materials are expressed.

Because of the complicated coupling effects, due to $A_{16}$, $A_{26}$, $B_{ij}$ etc., the behavior of a laminated composite can best be described using the A, B, and D matrices, or their equivalents, without specific references to engineering constants. For example, the effective in-plane

shear modulus of a laminated anisotropic material may be $A_{66}$, $1/A^{*}_{66}$, or $1/A'_{66}$. In general, their numerical values are different. Which component is being measured depends on the loading condition of the test. For a panel shear test, where only $N_6 \neq 0$, and $N_1 = N_2 = M_1 = 0$, $A'_{66}$ is being measured. In a circular tube under torsion, the loading conditions are as follows: $N_6 \neq 0$, and $N_1 = N_2 = k_1 = 0$; then $A^{*}_{66}$ is being measured. Component $A_{66}$ can be measured directly if the loading conditions satisfy $e_6 \neq 0$, and $e_1 = e_2 = k_1 = 0$. These conditions may be difficult to achieve.

In the case of uniaxial tension of a laminated composite, say, $N_1 \neq 0$, $A'_{11}$ and $A'_{12}$ are the components that govern the axial and transverse strains. If a circular tube is loaded along its generator, $A^{*}_{11}$ and $A^{*}_{12}$ are related to the axial (longitudinal) and circumferential strains. In general, $A'_{11} \neq A^{*}_{11}$ and $A'_{12} \neq A^{*}_{12}$. Components $A_{11}$ and $A_{12}$ are difficult to measure directly because specified strains rather than stresses must be imposed. This is anologous to components $C_{11}$ being more difficult to measure than $S_{11}$.

To avoid confusion, all properties of a laminated anisotropic composite should be reported in terms of the component of A, B, and D matrices. The use of engineering constants should be avoided.

# SECTION V

## STRENGTH OF COMPOSITE MATERIALS

The strength of composite materials, whether unidirectional or laminated, is considerably more complicated than the elastic moduli. No unified treatment comparable to that of the elastic behavior is available.

Three common strength theories can be readily applied to the composite materials. They are the maximum stress, maximum strain, and maximum distortional work theories.

In the unidirectional composite, which is assumed to be orthotropic and quasi-homogeneous, the maximum stress theory is expressed by three inequalities:

$$\begin{aligned} \sigma_x &\leq X \\ \sigma_y &\leq Y \\ \sigma_s &\leq S \end{aligned} \tag{248}$$

where $\sigma_x$, $\sigma_y$, and $\sigma_s$ are the stress components (a state of plane stress is assumed) relative to the material symmetry axes; X = axial strength (along the fibers); Y = transverse strength; and S = shear strength. Failure of the composite is induced when one or more of the equalities in Equation 248 are satisfied.

The maximum strain theory can also be expressed in terms of three inequalities:

$$\begin{aligned} e_x &\leq X_e \\ e_y &\leq Y_e \\ e_s &\leq S_e \end{aligned} \tag{249}$$

where $e_x$, $e_y$, and $e_s$ are the strain components: $X_e$ = ultimate axial strain; $Y_e$ = ultimate transverse strain; and $S_e$ = ultimate shear strain. According to this theory, failure is induced when one or more equalities are satisfied.

The distortional work theory, in plane stress, can be expressed by

$$\left(\frac{\sigma_x}{X}\right) - \frac{\sigma_x \sigma_y}{XY} + \left(\frac{\sigma_y}{Y}\right)^2 + \left(\frac{\sigma_s}{S}\right)^2 = 1 \tag{250}$$

This theory can be represented by a smooth quadratic surface in the stress space. The maximum stress or maximum strain theory can be represented by a cube in stress or strain space. The principal difference between the distortional work and the maximum stress or strain theories lies in the existence interaction among the anisotropic strengths. The distortional work theory contains a high level of interaction, whereas the maximum stress or strain theory assumes no interaction. Based on available strength data obtained from glass-epoxy composites, the distortional work theory appears to be more accurate than maximum stress or strain theory.

The strength of a laminated composite can be predicted if the elastic moduli and the strengths of each constituent layer are known. The strain components at each location can be obtained from Equation 241, and the stress, from Equation 242. Once these components are known, they can be substituted into appropriate strength theories.

In a laminated composite, not all layers will fail simultaneously. As one or several of the layers have failed, the layers which are still intact may be able to sustain the existing load. The shifting of the stress distribution within a laminated composite may cause an abrupt change in slope in the stress-strain curve of the composite. This is often referred to as the knee. The ultimate strength of the composite is reached where the still intact layers cannot carry the existing load. This strength analysis of a laminated composite agrees reasonably well with available data obtained from glass-epoxy laminated composite behavior.

A considerable amount of work is still needed for a basic understanding of the strength of composite materials. The theories just described are purely phenomenological, and no specific reference to the actual mechanisms of deformation and fracture is made.

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13 ABSTRACT


**This report covers some of the principles of the mechanics relevant to the description of composite materials. The contents of these notes may provide useful information for the understanding of current publications and reports related to composite materials.**

**Emphasis is placed on the use of indicial notation and operations. The rules governing the use of the contracted notation are also outlined. The generalized Hooke's law and its transformation properties, material symmetries, and engineering constants are also discussed. The plane strain and plane stress problems are discussed in detail. Finally, the elastic moduli of laminated anisotropic materials, and the strength of both unidirectional and laminated composites are covered.**


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